A

TREATISE

OF

Angular Sections.

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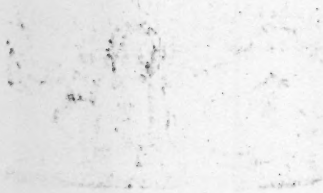


A TREATISE OF Angular Sections.

CHAP. I.

Of the Duplication and Biſection of an ARCH *or* ANGLE.

I. LET the Chord (or Subtenſe) of an Arch propoſed, be called A, (or E;) of the Double, B; of the Treble, C; of the Quadruple, D; of the Quintruple, F; *&c.* The Radius, R; the Diameter, 2R. (But ſometimes we ſhall give the name of the Subtenſe A, E, *&c.* to the Arch whoſe Subtenſe it is; yet with that care, as not to be liable to a miſtake.)

II. Where the Subtenſe of an Arch is A; let the Verſed ſine be V: (where that is E, let this be U.) Which drawn into (or Multiplied by) the remainder of the Diameter (2R—V) makes 2RV—Vq, the Square of the Right-ſine: (this Sine being a Mean-proportional between the Segments of the Diameter on which it ſtands erect, by 13 *el.* 6.) That is, (Q:¼B:) the Square of (the Right-ſine, or) half the Subtenſe of the double Arch: That is, 2RV—Vq = Q:¼B: = ¼Bq. *Fig.* I.

III. If to this we add Vq (the Square of the Verſed-ſine,) it makes 2RV = (¼Bq + Vq =) Aq. (And, by the ſame reaſon, 2RU = Eq.) That is,

IV. *The Subtenſe of an Arch, is a Mean Proportional between the Diameter and the Verſed-ſine.*

V. Again, becauſe 2RV = Aq, therefore (dividing both by 2R,) $\frac{Aq}{2R} = V$: And (the Square thereof) $\frac{Aqq}{4Rq} = Vq$: Which ſubtracted from Aq, leaves the Square of the Right-ſine, $Aq - \frac{Aqq}{4Rq} = Q:¼B$. (And, in like manner, $\frac{Eq}{2R} = U$, and $\frac{Eqq}{4Rq} = Uq$, and $Eq - \frac{Eqq}{4Rq} = Q:¼B$. That is,

VI. *If from the Square of the Subtenſe, we take its Biquadrate divided by the Square of the Diameter; the Remainder is equal to the Square of the Right-ſine: And the Square-root of that Remainder, to the Sine it ſelf: And, the double of this, to the Subtenſe of the double Arch.*

VII. Accordingly, becauſe $Aq-\frac{Aqq}{4Rq}=\frac{1}{4}Bq$, therefore (its Quadruple) $4Aq-\frac{Aqq}{Rq}=Bq$; and $\surd:4Aq-\frac{Aqq}{Rq}:=B$. (And in like manner, $\surd:4Eq-\frac{Eqq}{Rq}:=B$.) That is,

VIII. *If from Four-times the Square of a Subtenſe, are taken its Biquadrate divided by the Square of the Radius; the Remainder is the Square of the Subtenſe of the double Arch: And, the Quadratick Root of that Remainder, is the Subtenſe it ſelf.*

IX. But $\surd:44q-\frac{Aqq}{Rq}:(=B)=\sqrt{\frac{4AqRq-Aqq}{Rq}}=\frac{A}{R}\surd:4Rq-Aq:=\frac{A}{R}E=\frac{AE}{R}$. That is,

X. *The Rect-angle of the Subtenſes of an Arch, and of its Remainder to a Semicircle, divided by the Radius; is equal to the Subtenſe of the double Arch.*

XI. Becauſe $\frac{AE}{R}=B$; therefore $AE=RB=2R\times\frac{1}{2}B$: And, $R.A::B.B$: And therefore, (becauſe AE contain a Right-angle, as being an Angle in a Semicircle.)

XII. *In a Right-angled Triangle, the Rect-angle of the two Legs containing the Right-angle, is equal to that of the Hypothenuſe, and the Perpendicular from the Right-angle therenpon.* And,

XIII. *As the Radius, to the Subtenſe of an Arch; ſo the Subtenſe of its Remainder to a Semicircle, is to that of the double Arch.*

XIV. Becauſe B, the Subtenſe of a double Arch, doth indifferently ſubtend the two Segments which compleat the whole Circumference; and, conſequently, the half of either may be the ſingle Arch of this double: It is therefore neceſſary that this Equation have two (Affirmative) Roots; the greater of which we will call A; and the leſſer E: And therefore $\surd:4Aq-\frac{Aqq}{Rq}:=B=\surd:4Eq-\frac{Eqq}{Rq}$. That is,

XV. *Any Arch, and its Remainder to a Semicircumference, (as alſo its exceſs above a Semicircumference, and either of them increaſed by one or more Semicircumferences,) will have the ſame Subtenſe of the double Arch.* For in all theſe Caſes, the Subtenſe of the ſingle Arch will be either A or E.

XVI. Becauſe $\surd:4Aq-\frac{Aqq}{Rq}:(=B)=\surd:4Eq-\frac{Eqq}{Rq}:$ And therefore, $4Aq-\frac{Aqq}{Rq}(=Bq)=4Eq-\frac{Eqq}{Rq}$; and $4AqRq-Aqq(=BqRq)=4EqRq-Eqq$: Therefore (by Tranſpoſition) $4AqRq-4EqRq=Aqq-Eqq$; and (dividing both by $Aq-Eq$,) $4Rq=\left(\frac{Aqq-Eqq}{Aq-Eq}=\right)Aq+Eq$. That is,

XVII. *The*

XVII. *The Square of the Diameter, is equal to the difference of the Biquadrates of the Subtenſes of two Arches, (which together complete a Semicircumference) divided by the difference of their Squares: And this alſo, equal to the ſum of the Squares of thoſe Subtenſes.* That is, (becauſe A E contain a Right-angle.)

XVIII. *In a Right-angled Triangle, the Square of the Hypothenuſe* (4 R q) *is equal to the Squares of the ſides containing the Right-angle.* (A q + E q.)

XIX. Or thus, Becauſe B is the common Subtenſe to two Segments, which together complete the whole Circumference; and therefore the half of both, complete the Semicircumference: If therefore in a Circle (according to *Ptolemy*'s *Lemma*) a Trapezium be inſcribed, whoſe oppoſite ſides are A, A; and E, E: The Diagonals will be Diameters, that is, 2 R: And, conſequently, 4 R q = A q + E q; as before. *Fig.* II.

XX. Hence therefore, *The Radius* (R) *with the Subtenſe of an Arch* (A or E) *being given; we have thence the Subtenſe of the double Arch,* B: (which is the *Duplication of an Arch* or *Angle.*) For, R, A, being given; we have $E = \sqrt{:4Rq - Aq}$. (or R, E, being given, we have $A = 4Rq - Eq$:) And, having R, A, E; we have $B = \frac{AE}{R}$; by § 9.

XXI. *The Radius* R, *with* B *the Subtenſe of the double Arch, being given; we have thence the Subtenſe of the ſingle Arch,* A or E. (which is the *Biſection of an Arch* or *Angle.*) For, by § 14, $\sqrt{:4Aq - \frac{Aqq}{Rq}:} = B = \sqrt{:4Eq - \frac{Eqq}{Rq}:}$ And therefore $4RqAq - Aqq\ (= RqBq) = 4RqEq - Eqq$. And the Roots of this Equation, $2Rq \pm \sqrt{:4Rqq - RqBq:} = 2Rq \pm R\sqrt{:4Rq - Bq:} = Aq$, or Eq. And, the Quadratick Root of this, is A, or E.

XXII. Hence alſo we have an eaſie *Method, for a Geometrical Conſtruction for the Reſolution of ſuch Biquadratick Equations; or Quadratick Equations of a Plain Root, wherein the Higheſt Power is Negative.* (Underſtand it in Mr. *Oughtred*'s Language: Who puts the Abſolute Quantity, Affirmative; and by it ſelf; and the reſt of the Equation all on the other ſide.) Suppoſe, $RqBq = 4RqAq - Aqq$, or (putting $P = \frac{1}{2}B$,) $4RqPq = 4RqAq - Aqq$. For, dividing the Abſolute term RqBq, or 4RqPq, by the Co-efficient of the middle term 4Rq, the Reſult is $\frac{1}{4}Bq$, or Pq; and its Root $\frac{1}{2}$ B or P. Which being ſet Perpendicular on a Diameter equal to 2 R (the Square Root of that Co-efficient:) a ſtreight Line from the top of it, Parallel to that Diameter, will (if the Equation be not impoſſible) cut the Circle, or at leaſt touch it: From which Point of Section or Contact, two ſtreight-lines drawn to the ends of the Diameter, are A, and E, the two Roots of that (ambiguous) Biquadratick Equation, (or, if we call it a Quadratick of a Plain-root, the Root of the Plain-root of ſuch Quadratick Equation.) *Fig.* III.

XXIII. And this Conſtruction, is the ſame with the Reſolution of this Problem; *In a Right-angled Triangle, the Hypothenuſe being given, and a Perpendicular from the Right-angle thereupon, to find the other ſides;* (and, if need be, the Angles, the Segments of the Hypothenuſe, and the Area of the Triangle $\frac{1}{2}$RB or PR.)

XXIV. Or thus: Having R and B, (as at § 22.) with the Radius R deſcribe a Circle; and therein inſcribe the Chord B; and another on the middle hereof at Right-angles: (which will therefore biſect that, and be a Diameter:) And, from both ends of this, to either end of B, draw the Lines A, E; as before. And this Conſtruction is better than the former, becauſe of the uncertainty of the preciſe Point of Contact or Section, in caſe the Section be ſomewhat Oblique.

XXV. Now

XXV. Now if it be desired, in like manner, to give a like Construction, in Case of *such Biquadratick Equations* (or Quadraticks of a Plain-root) *where the highest Power is Affirmative*; (though that be here a Digression, as in all the rest that follow, to § 35.) It is thus: Suppose the Equations $Aqq - VqAq = VqEq = Pqq + VqPq$: Whose (Affirmative) Roots are Aq, and Pq; (and therefore Vq, $VqEq$, and consequently Eq, are known Quantities:) Therefore (by Transposition) $Aqq - VqAq + VqPq$; and (dividing by $Aq + Pq$,) $Aq - Pq \left(= \frac{Aqq - Pqq}{Aq + Pq}\right) = Vq$. And therefore $Aq - Vq = Pq$; and $Pq + Vq = Aq$: And (by Multiplication) $Aqq - VqAq = AqPq = Pqq + VqPq = VqEq$.

$$\begin{array}{l} Aq - Vq = Pq \\ \times Aq \end{array} \qquad \begin{array}{l} Aq = Pq + Vq \\ \times Pq \end{array}$$

$$Aqq - VqAq = AqPq == Pqq + VqPq = VqEq.$$

XXVI. The Equation therefore proposed (dividing all by Vq) comes to this, $\frac{Aqq}{Vq} - Aq = \frac{AqPq}{Vq} = Eq = \frac{Pqq}{Vq} + Pq$. That is, $\frac{Aqq}{Vq} - V \times \frac{Aq}{V} = \frac{AqPq}{Vq} = Eq = \frac{Pqq}{Vq} + V \times \frac{Pq}{V}$. Whose Roots are $\frac{Aq}{V}$, and $\frac{Pq}{V}$. Namely, $\surd : Eq + \frac{1}{4}Vq : + \frac{1}{2}V = \frac{Aq}{V}$. And $\surd : Eq + \frac{1}{4}Vq : - \frac{1}{2}V = \frac{Pq}{V}$. And these Multiplied into V (a known Quantity) make Aq, and Pq: Namely, $V\surd : Eq + \frac{1}{4}Vq : + \frac{1}{2}Vq = Aq$. And $V\surd : Eq + \frac{1}{4}Vq : - \frac{1}{2}Vq = Pq$. And consequently, A is a mean Proportional between V and $\surd : Eq + \frac{1}{4}Vq : + \frac{1}{2}V$. And P, a mean Proportional between V and $\surd : Eq + \frac{1}{4}Vq : - \frac{1}{2}V$. Therefore,

Fig. IV. V XXVII. An Equation being proposed in one of these Forms, $Aqq - VqAq = VqEq = Pqq + VqPq$: The absolute term ($VqEq$) being divided by the Co-efficient of the middle term (Vq;) the quantity resulting is (Eq) whose Square-root (E,) set Perpendicular on the end of a streight Line equal to (V) the Square-root of the Co-efficient; which we may suppose the Diameter of a Circle, to which that Perpendicular is a Tangent: On the same Center with this Circle (and on the same Diameter continued) by the Top of that Perpendicular, draw a second Circle. The Diameter of this second Circle, is, by that Perpendicular (E) cut into two Segments, which are the Roots of these Equations. That is, $U = \frac{Pq}{V}$; and $V + U = \frac{Aq}{V}$.

XXVIII. Or, (without drawing that second Circle,) from the Top of that Perpendicular (in a streight Line through the Center of the first, which will cut the Circumference in two Points,) to the first Section, is $U = \frac{Pq}{V}$; to the second, $U + V = \frac{Aq}{V}$.

XXIX. These two Roots, Multiplied one into the other, become equal to the Absolute quantity, $\frac{AqPq}{Vq} = Eq$. And Multiplied into V, become Aq, Pq: Or, thus, P is a mean Proportional between V and U; and A, between V and $V + U$: Or thus, P is a mean Proportional between V and $U = \frac{Pq}{V}$; and (because by § 25. $Aq = Pq + Vq$,) A is the Hypothenuse (in a Right-angled Triangle) to the Legs P, V. And this is no contemptible Method, *For the resolving Quadratick Equations of a Plain-root, wherein the highest term is Affirmative.* The whole Geometrick Construction, is clear enough from the Figures adjoined; where yet the Circles, (for the most part) serve rather for the Demonstration, than the Construction.

XXX. Again,

XXX. Again, (by the ſame § 25.) $Eq = \frac{Pqq}{Vq} + Pq = Uq + Pq$. And therefore A and E, are alſo the Legs of a Right-angled Triangle, whoſe Hypothenuſe is $V + U$: Which, by P (a Perpendicular on it from the Right-angle) is cut into thoſe two Segments.

XXXI. From the ſame Conſtruction therefore, we have alſo the Geometrical Conſtruction of this Problem; *In a Right-angled Triangle, having one of the Legs* E, *with the farther Segment of the Hypothenuſe* V, *to find the other Segment*; (and ſo the whole $V + U$; and the Perpendicular P; and the other Leg A; and the whoſe Triangle.)

XXXII. We have thence alſo this Analogy; V . P :: A ($= \surd : Pq + Vq$:) . E. And V . A :: P ($= \surd : Aq - Vq$:) . E. Or thus, $Vq . Pq :: Pq + Vq . Eq$. And $Vq . Aq :: Aq - Vq . Eq$.

XXXIII. If therefore we make V the Radius of a Circle; then is A the Secant; P, the Tangent; E a Parallel to the Right-ſine (in contrary poſition) from the end of the Secant to the Diameter produced. If we make A the Radius; then is P the Right-ſine, and E the Tangent of the ſame Arch; and V, the Sine of the Complement, or Difference between the Radius and Verſed ſine. From hence therefore, *Fig.* VI.

XXXIV. *The Tangent* E, *and Sine of the Complement* V, *being given; we have the Right-ſine* P, *and the Radius* A. (But, § 25, and all hitherto, is a Digreſſion.)

XXXV. If in a Semicircle on the Diameter 2 R, we inſcribe B the Subtenſe of a double Arch: A Perpendicular on the middle Point hereof, will cut the Arch of that Semicircle into two Segments, (whoſe Subtenſes are A, E;) either of which is a ſingle Arch, to the double whereof, B is a Subtenſe. This, as to E, is evident from 4 è 1, and 28 è 3: And, as to A, from § 15 of this. *Fig.* VII.

XXXVI. But alſo (by the ſame reaſon,) the Arch ß (the difference of the Arches A, E;) and B (the double of either,) will (if doubled) have the ſame Subtenſe of their double Arch. That is, *The double of the double* (of either) *and the double of their difference, will have the ſame Subtenſe.*

XXXVII. If an Arch to be doubled, be juſt a third part of the Circumference; the Subtenſe of the double, is equal to that of the ſingle Arch. (For the ſame Subtenſe, which on one ſide ſubtends two Trients, doth on the other ſide ſubtend but one.) That is, by § 7, $4Aq - \frac{Aqq}{Rq} (= Bq) = Aq$. And therefore (by Tranſpoſition) $3Aq = \frac{Aqq}{Rq}$, and $3Rq = Aq$. That is, *Fig.* VIII.

XXXVIII. *The Square of the Subtenſe to a Trient of the Circumference* (or of the ſide of an Equilater Triangle inſcribed) *is equal to three Squares of the Radius.*

XXXIX. Again, the ſame being the Subtenſe of the double Trient, and of the double Sextant, (for a Trient and a Sextant compleat the half, $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$) the Square of the Subtenſe of a Sextant, (Eq,) is the difference of the Squares of that of the Trient, and (the Diameter or) that of the Semicircumference. That is, $4Rq - Aq = Eq$; that is, (by § *preced.*) $4Rq - 3Rq = Rq = Eq$: And, $E = R$. That is,

XL. *The Subtenſe of a Sextant* (or ſide of the inſcribed Equilater Hexagon) *is equal to the Radius.*

CHAP.

CHAP. II.

Of the Triplication and Triſection of an ARCH *or* ANGLE.

Fig. IX. I. IF in a Circle, be inſcribed a Quadrilater, whoſe three ſides are A, A, A, (Subtenſes of a ſingle Arch) and the fourth C, (the Subtenſe of the Triple Arch:) the Diagonals are B, B, the Subtenſe of the double;) as is evident. But it is evident alſo, that (in this Caſe) A is leſs than a Trient of the whole Circumference.

II. And therefore (the Rect-angle of the Diagonals being equal to the two Rect-angles of the oppoſite ſides,) $Bq = Aq + AC$; and therefore $Bq - Aq = AC$, and $\frac{Bq - Aq}{A} = \frac{Bq}{A} - A = C$. That is,

III. *The Square of the Subtenſe of the double Arch; is equal to the Square of the Subtenſe of the Single Arch (leſs than a Trient of the Circumference) and the Rect-angle of the Subtenſes of the Sngle and Treble Arch.* And therefore,

IV. *The Square of the Subtenſe of the double Arch, wanting the Square of the Subtenſe of the ſingle Arch, (being leſs than a Trient,) is equal to the Rect-angle of the Subtenſes of the ſingle and Treble Arch.* And conſequently,

V. *If the Square of the Subtenſe of the double Arch, wanting the Square of the Subtenſe of the ſingle Arch (leſs than a Trient,) be divided by the Subtenſe of the ſingle Arch; the Reſult is the Subtenſe of the Triple Arch.*

VI. Becauſe that (by § 2.) $\frac{Bq - Aq}{A} = C$; and, that $B + A$ into $B - A$ is equal to $Bq - Aq$: (as will appear by Multiplication:) Therefore, $A . B + A :: B - A . C$. That is,

VII. *As the Subtenſe of a ſingle Arch (leſs than a Trient) to the ſum of the Subtenſes of the ſingle and double Arch; ſo is the Exceſs of that of the double above that of the ſingle, to the Subtenſe of the Triple.*

VIII. Again, becauſe (by § 7, *of the precedent Chapter,*) $Bq = 4Aq - \frac{Aqq}{Rq}$: Therefore, $Bq - Aq\ (= AC) = 3Aq - \frac{Aqq}{Rq}$: And therefore $\frac{Bq - Aq}{A} = 3A - \frac{Ac}{Rq} = C$. That is,

IX. *The Triple of the Subtenſe of an Arch (leſs than a Trient,) wanting the Cube thereof, divided by the Square of the Radius; is equal to the Subtenſe of the Triple Arch.*

X. But, becauſe the ſame Subtenſe C, ſubtends alſo to another Segment of the ſame Circle; the Subtenſe of whoſe Trient we ſhall call E: Therefore $3A - \frac{Ac}{Rq} = C = 3E - \frac{Ec}{Rq}$.

XI. And becauſe the three Arches A, A, A; and the three Arches E, E, E; complete the whole Circumference: (as is evident;) Therefore, once A, and once E, complete a Trient or third part thereof. Therefore,

XII. *An*

XII. *An Arch leſs than the Trient of a Circumference, and the Reſidue of that Trient,* (A, and E,) *have the ſame Subtenſe of their Triple Arch.* Fig. IX.

XIII. Again, becauſe (as is ſhewed already) $3A - \frac{Ac}{Rq} = 3E - \frac{Ec}{Rq}$; and therefore $3RqA - Ac = 3RqE - Ec$; and $3RqA - 3RqE = Ac - Ec$: Therefore, (dividing both by $A - E$,) $3Rq\left(= \frac{3RqA - 3RqE}{A - E}\right) = \frac{Ac - Ec}{A - E}$ $= Aq + AE + Eq$. (As will appear upon dividing $Ac - Ec$ by $A - E$; or Multiplying $A - E$ into $Aq + AE + Eq$.)

XIV. But (by § 37, 38, *Chap. preced.*) $3Rq$ is the Square of the Subtenſe of a Trient; that is (by § 11 of this) of the ſum of the Arches A and E. Therefore,

XV. *The Square of the Subtenſe of the Trient of the Circumference of a Circle,* (or three Squares of the Radius,) *is equal to the Squares of the Subtenſes of any two Arches completing that Trient, and the Rect-angle of them.* That is, (putting T for the Subtenſe of a Trient) $Tq\ (= 3Rq) = Aq + AE + Aq$.

XVI. But the Angle which AE contain, (as being an Angle in the Trient of a Circle, or inſiſting on two Trients,) is an Angle of 120 Degrees. And therefore (by § 15.)

XVII. *In a Right-lined Triangle, one of whoſe Angles is* 120 *Degrees; the Square of the Subtenſe to that Angle, is equal to the two Squares of the ſides containing it, and a Rect-angle of thoſe ſides.* (For, if ſuch Triangle be inſcribed in a Circle, the Baſe of that Triangle, will be the ſubtendent of a Trient in ſuch Circle; or $\sqrt{3Rq}$.)

XVIII. If a Quadrilater be inſcribed in a Circle, three of whoſe ſides are A, E, A, (or E, A, E,) and the fourth Z: Each of the Diagonals (by § 11.) is T, the Subtenſe of a Trient. And therefore (by § 13, 14, 15,) $ZE + Aq$ $(= ZA + Eq) = Tq = 3Rq = Aq + AE + Eq$. And, conſequently, $ZE = AE + Eq$, and $ZA = Aq + AE$; and therefore, $Z = A + E$. And therefore, Fig. X.

XIX. *If to the Aggregate of two Arches* A, E, *(completing a Trient,) be added a third equal to either of them;* Z, *the Subtenſe of the Aggregate of all the three, is equal to the ſum of the Subtenſes of thoſe two.* That is, $Z = A + E$.

XX. But the ſame Chord Z, doth ſubtend, on the one ſide, to a Trient increaſed by the Arch A; and, on the other ſide, to a Trient increaſed by the Arch E; (as is evident;) That is, to an Arch which doth as much exceed a Trient (or want of two Trients,) as the Arch A or E wants of a Trient. Therefore,

XXI. *The Aggregate of the Subtenſes of two Arches, which together make up a Trient; is equal to the Subtenſe of another Arch which doth as much exceed a Trient, (or want of two Trients,) as either of thoſe two wants of a Trient.*

XXII. The ſame will in like manner be inferred, if we inſcribe a Quadrilater whoſe oppoſite ſides are A, T, and E, T; and the Diagonals TZ. For then $TA + TE = TZ$; and therefore $A + E = Z$, as before. Fig. XI.

Fig. XI. XXIII. But if either of the Arches to which Z ſubtendeth (greater than a Trient, and leſs than two Trients) be Tripled; the Subtenſe of this Triple, is the ſame with that of the Triple of A or E. For the Triple of an Arch greater than a Trient, is equal to one whole Circumference with the Triple of that Exceſs. (For the Triple of $\frac{1}{3} + A$, is $1 + 3A$.) Now becauſe, when we have once gone round the whole Circumference, we are juſt there where at firſt we began; this therefore (as to this Point) is as nothing; and the whole diſtance to be acquired is but the Triple of ſuch Exceſs; and juſt the ſame as if onely this Exceſs had been thrice taken.

Fig. XII. XXIV. As for Example: If the Arch ſubtended by Z, be $\beta\gamma\delta$, (that is, a Trient increaſed by the Arch E;) and to this we add a ſecond equal to it, $\delta\zeta\theta$; the Aggregate $\beta\gamma\delta\zeta\theta$, is the double Arch, and the Subtenſe thereof is B, or $\beta\theta$, (which is alſo the Subtenſe of the Difference of the Arches A, E:) and if to theſe two, we add a third equal to either of them $\theta\gamma\varkappa$; then is $\beta\gamma\delta\zeta\theta\gamma\varkappa$, the Triple of the Arch firſt propoſed; and the Subtenſe hereof (that is, the Streight-line which joyns the beginning and the end of this Triple Arch) is $\beta\varkappa = C$; the very ſame which ſubtends the Triple of E.

XXV. And juſt the ſame would come to paſs, if for the firſt Arch we take $\beta\theta\zeta\delta$ (that is, a Trient increaſed by A,) to which Z is a ſubtendent likewiſe. For, taking a ſecond equal to it $\delta\varkappa\gamma\beta\theta$; the Aggregate $\beta\theta\zeta\delta\varkappa\gamma\beta\theta$ (more than one entire Circumference) is the double Arch, and the Subtenſe thereof B as before: And if to theſe two we add a third equal to either, $\theta\zeta\delta\varkappa$; the Triple Arch is $\beta\theta\zeta\delta\varkappa\gamma\beta\theta\zeta\delta\varkappa$; and the Subtenſe hereof (as before) $\beta\varkappa$ or C; the ſame with the ſubtendent of the Triple of A. And therefore,

XXVI. *The Triple of an Arch greater than a Trient, hath the ſame Subtenſe with the Triple of its Exceſs above a Trient.* And the ſame (for the ſame reaſon) holds in Arches greater than 2, 3, or more Trients.

XXVII. But note here, that, in this caſe; That is, if the Arch to be Tripled be greater than a Trient, but leſs than two Trients, (for if more than two Trients, but leſs than the whole Circumference, it is the ſame as if it were leſs than a Trient;) the Subtenſe of the double is leſs than that of the ſingle. For, in ſuch caſe, the Arch will differ from that of a Semicircle (either in Exceſs or in Defect) by leſs than $\frac{1}{6}$ of the whole Circumference. Let it be X. If therefore $\frac{1}{2} \pm X$ be the ſingle Arch, the double will be $1 \pm 2X$; and the Subtenſe thereof (whether greater or leſs than one entire revolution) will be the ſame with that of $2X$: And therefore (X being leſs than $\frac{1}{6}$,) $2X$ will require a leſs Subtenſe than that of $\frac{1}{3} \pm X$; that being leſs than the Subtenſe of a Trient, but this greater than it. And the like is to be underſtood in other caſes of like nature.

XXVIII. Suppoſing therefore, as before, $\frac{Bq - Aq}{A} = C$, or $\frac{Bq - Zq}{Z} = C$; C muſt in this caſe be a Negative quantity: Or, if we put C Affirmative, then muſt Z be Negative, (or leſs than nothing:) For $Bq - Zq$ (where a greater quantity is to be ſubtended from a leſs) muſt needs be Negative; that is $Bq - Zq = Zc$; where Zc being a Negative, either Z or C muſt be ſo too, or elſe (putting all Affirmative) $Zq - Bq = Zc$, and $Zq = Bq + ZC$.

XXIX. Which is evident alſo from the Diagram; where, for this reaſon, ZZ become Diagonals; and both BB, and ZC, oppoſite ſides. And therefore $Zq - Bq = ZC$, or $Zq = Bq + Zc$; and $\frac{Zq - Bq}{Z} = Z - \frac{Bq}{Z} = C$. That is,

XXX. *The*

XXX. *The Square of the Subtenſe of a ſingle Arch, greater than a Trient, but leſs than two Trients; is equal to the Square of the Subtenſe of the double Arch, together with a Rect-angle of the Subtenſes of the ſingle and Triple Arch.* And, Fig. XII.

XXXI. *The Square of the Subtenſe of a ſingle Arch (greater than a Trient, but leſs than two Trients,) wanting the Square of the Subtenſe of the double Arch; is equal to the Rectangle of the Subtenſes of the ſingle and Triple Arch.* And therefore,

XXXII. *If the Square of the Subtenſe of a ſingle Arch (greater than a Trient, but leſs than two Trients,) wanting the Square of the Subtenſe of the double Arch, be divided by that of the ſingle, the Reſult is the Subtenſe of the Triple Arch:* (Or, if divided by that of the Triple, the Reſult is that of the ſingle.) Or,

XXXIII. *If from the Subtenſe of a ſingle Arch (greater than a Trient but leſs than two Trients,) we Subtract the Square of the Subtenſe of the double Arch divided by that of the ſingle; the Remainder is equal to the Subtenſe of the Triple.*

XXXIV. But, becauſe of $\frac{Zq-Bq}{Z} = C$, or $Z) Zq-Bq (C$; and $Zq-Bq = Z+B$ into $Z-B$: We have thence this Analogy, $Z . Z+B :: Z-B . C$. That is,

XXXV. *As the Subtenſe of a ſingle Arch (greater than a Trient but leſs than two Trients,) to the Aggregate of the Subtenſes of the ſingle and double; ſo is that of the ſingle wanting that of the double to that of the Triple.*

XXXVI. Now becauſe (as we have ſhewed) $Zq-Bq=ZC$; and (by § 7, 8, *Chap. preced.*) $Bq=4Zq-\frac{Zqq}{Rq}$: Therefore $Zq-Bq(=Zq-4Zq+\frac{Zqq}{Rq}) = \frac{Zqq}{Rq}-3Zq$: And $\frac{Zq-Bq}{Z} = \frac{Zc}{Rq}-3Z=C$. That is,

XXXVII. *If from the Cube of the Subtenſe of a ſingle Arch (greater than a Trient but leſs than two Trients) divided by the Square of the Radius; we ſubtract the Triple of that Subtenſe: The Remainder is equal to the Subtenſe of the Triple Arch.*

XXXVIII. If the Arch to be Tripled be greater than two Trients, it is the ſame as if it were leſs than one Trient. (For the Reſidue of the whole, to which it alſo ſubtends, is then leſs than a Trient.) And therefore the ſame Chord (ſuppoſe A or E) ſubtends as well to an Arch greater than two Trients, as to one leſs than one Trient.

XXXIX. If the Arch to be Tripled be equal to a Trient; it is indifferent to whether of the two caſes it be referred, (that of the greater, or that of the leſſer, than a Trient,) and the ſame happens if it be ſuppoſed equal to two or more Trients, or to one or more intire revolutions.

XL. If the Arch to be Tripled be greater than one or more intire revolutions; its Subtenſe is the ſame with that of its Exceſs above thoſe intire revolutions, and to be conſidered in like manner, which things are evident and need no further demonſtration.

XLI. Now what hath been ſeverally delivered concerning the Triplication of an Arch or Angle leſs than a Trient; and of one greater than a Trient, but leſs than two Trients; (to one of which caſes every Arch may be referred, as is already ſhewed;) we may thus, jointly put together.

Fig. XII. XLII. *The Difference of the Squares of the Subtenſes of the ſingle and double Arch (whether ſoever of them be the greater,) is equal to the Rect-angle of the Subtenſes of the ſingle and Triple.* (by § 4 and 31.) That is, $Bq-Aq=AC$, and $Zq-Bq=ZC$. And therefore,

XLIII. *If the Difference of the Squares of the Subtenſes of the ſingle and double Arch, be divided by the Subtenſe of the ſingle; it gives that of the Triple:* If, by that of the Triple, it gives that of the ſingle. (by § 5, 32.) That is, $\frac{Bq-Aq}{A}=C=\frac{Zq-Bq}{Z}$. And $\frac{Bq-Aq}{C}=A$. And $\frac{Zq-Bq}{C}=Z$. And,

XLIV. *As the Subtenſe of the ſingle Arch, to the Aggregate of the Subtenſes of the ſingle and double; ſo is the Difference of theſe Subtenſes, to the Subtenſe of the Triple.* (by § 7, 35.) That is, $A \cdot B+A :: B-A \cdot C$. And $Z \cdot Z+B :: Z-B \cdot C$.

XLV. Now for as much as (by § 12, 26.) the Three Arches A, E, Z, if Tripled, will have the ſame Subtenſe of the Triple Arch C: 'Tis thence manifeſt, that *ſuch Equation as this (which concerns the Triſection of an Arch,)* $3O \sim \frac{Oc}{Rq}=C$, *muſt have in all Three Roots*, as A, E, Z: (For every of theſe, upon ſuch Triplication, will have the Subtenſe of the Triple Arch, C:) Yet ſo; that, *where* A *and* E *are Affirmative Roots*, Z *is a Negative*; and contrarywiſe, where this is Affirmative, thoſe be Negative. That is, in this Equation, $3O-\frac{Oc}{Rq}=C$; the Roots are $+A, +E, -Z$. But, in this, $\frac{Oc}{Rq}-3O=C$; the Roots are $-A, -E, +Z$. And therefore, $3A-\frac{Ac}{Rq}=3E-\frac{Ec}{Rq}=C=\frac{Zc}{Rq}-3Z$. And, conſequently, $3ARq-Ac=3ERq-Ec=CRq=Zc-3ZRq$.

XLVI. Since therefore $3ARq-Ac=Zc-3ZRq$; and conſequently (by Tranſpoſition) $3ZRq+3ARq=Zc+Ac$; it is alſo (dividing both ſides by $Z+A$,) $3Rq=\left(\frac{Zc+Ac}{Z+A}=\right)Zq-ZA+Aq$. (For $Z+A$ into $Zq-ZA+Aq$, is equal to $Zc+Ac$; as will appear by Multiplication; and contrarywiſe, if this be divided by either of thoſe, the Quotient will be the other of them, as will be found by Diviſion.) And in like manner; becauſe alſo $3ERq-Ec=Zc-3ZRq$, therefore $3ZRq+3ERq=Zc+Ec$; and $3Rq=\left(\frac{Zc+Ec}{Z+E}=\right)Zq-ZE+Eq$. That is,

XLVII. *Of two Arches, whereof the one exceeds the other by a Trient of the whole Circumference; or elſe, whereof the one doth as much exceed a Trient as the other wants of it; the Squares of the Subtenſes, wanting a Rect-angle of the ſame Subtenſes, are equal to the Square of the Subtenſe of a Trient*, or *Three times the Square of the Radius.* That is, $Zq-ZA+Aq=3Rq=Tq=Zq-ZE+Eq$.

XLVIII. Now the Angle contained by the Legs ZA, or ZE, (ſtanding on the Chord T,) is an Angle of 60 Degrees; (as being an Angle in the Circumference ſtanding on an Arch of 120 Degrees.) And therefore,

Fig. XIII. XLIX. *In a Right-lined Triangle, one of whoſe Angles is of* 60 *Degrees; the Square of the ſide oppoſite to this Angle, is equal to the two Squares of the ſides containing it, wanting the Rect-angle of the ſame ſides.* (For any ſuch Triangle may be thus inſcribed in a Circle:) That is, $Zq-ZA+Aq$, (or $Zq-ZE+Eq$,) $=Tq=3Rq$.

L. The

L. The ſame things (from § 45, *&c.*) may be thus otherwiſe inferred. *Fig.* XIII
Becauſe (by § 15) $Aq+AE+Eq=3Rq$, and (by § 18 or 21.) $Z=A+E$; therefore $Zq=Aq+2AE+Eq$, and $ZA=Aq+AE$, (and $ZE=AE+Eq$,) and therefore $Zq-ZA=AE+Eq$, (and $Zq-ZE=Aq+AE$;) and conſequently $Zq-ZA+Aq$ (or $Zq-ZE+Eq$,) $=Aq+AE+Eq=3Rq$. From whence the reſt are inferred as before.

$+Aq+2AE+Eq$	$=+Zq$	$+Aq+2AE+Eq$	$=+Zq$
$-Aq-AE$	$=-ZA$	$-AE-Eq$	$=-ZE$
$+Aq$	$=+Aq$	$+Eq$	$=+Eq$
$Aq+AE+Eq$	$=Zq-ZA+Aq$	$Aq+AE+Eq$	$=Zq-ZE+Eq.$

LI. Moreover, becauſe (as is before ſhewed) $\frac{Ac-Ec}{A-E}=Aq+AE+Eq=3Rq=Tq=\frac{Zc+Ac}{Z+A}=Zq-ZA+Aq=Zq-ZE+Eq=\frac{Zc+Ec}{Z+E}$: We may thence infer the following Theorems.

LII. *The Difference of the Cubes of two Legs containing an Angle of* 120 *degrees, divided by the Difference of thoſe Legs, is equal to the Square of the Baſe ſubtended to it.*

LIII. *But if it be an Angle of* 60 *degrees; the ſum of the Cubes of the Legs or ſides containing it, divided by the ſum of thoſe ſides, is equal to the Square of the Baſe.* Again,

LIV. *The Difference of the Legs containing an Angle of* 120 *degrees, Multiplied into the Square of the Baſe, is equal to the Difference of the Cubes of thoſe Legs.*

LV. *But if it be an Angle of* 60 *degrees; the ſum of the Legs Multiplied into the Square of the Baſe is equal to the ſum of the Cubes of thoſe Legs.*

LVI. Again, becauſe $3A-\frac{Ac}{Rq}$, (or $3E-\frac{Ec}{Rq}$,) $=C$; and $\frac{Zc}{Rq}-3Z=C$: Therefore $3A-C=\frac{Ac}{Rq}$, (and $3E-C=\frac{Ec}{Rq}$,) and $\frac{Zc}{Rq}=3Z+C$. And therefore,

LVII. *The Difference between the Triple of the Subtenſe of a ſingle Arch, leſs than a Trient; and, of the Subtenſe of the Triple of that Arch; is equal to the Cube of the Subtenſe of that ſingle Arch, divided by the Square of the Radius.* And conſequently, *That Difference Multiplied into the Square of the Radius, is equal to ſuch Cube.*

LVIII. *The ſum of the Triple of the Subtenſe of a ſingle Arch greater than a Trient, but leſs than two Trients, and of the Subtenſe of the Triple Arch; is equal to the Cube of the Subtenſe of that ſingle Arch divided by the Square of the Radius.* And conſequently, *That ſum Multiplied into the Square of the Radius, is equal to ſuch Cube.*

LIX. Becauſe (as is before ſhewed) $3A-\frac{Ac}{Rq}=3E-\frac{Ec}{Rq}=C=\frac{Zc}{Rq}-3Z$: Or, $3ARq-AC=3ERq-Ec=CRq=Zc-3ZRq$: Therefore *the Subtenſe of an Arch being given* (as A, E, or Z,) *together with the Radius* R; *we have thence the Subtenſe of the Triple Arch* C. Which is, *the Triplication of an Arch or Angle.*

LX. And

Fig. XIII LX. And, contrarywiſe; *The Radius of a Circle* R, *and the Subtenſe of the Triple Arch* C, *being given; we have thence the Subtenſe of the ſingle Arch*, (A, E, or Z,) *by reſolving ſuch a Cubick Equation.* Which is, *the Triſection of an Arch or Angle.* But the Geometrical effection thereof is not to be performed by Rule and Compaſs; without the help of a Conick Section, or ſome Line more Compounded.

LXI. But then, on the other ſide; *ſuch Cubick* Equations may be reſolved *by the Triſection of an Arch.* For, ſuppoſe a Cubick Equation of this Form, $3RqA - Ac$, (or $3RqE - Ec$,) $= RqC$; whoſe Root A (or E) is ſought. Now if R (the Square-root of a third part of the Co-efficient) be made the Radius of a Circle, (that is, $V\frac{3Rq}{3} := R$;) and therein be inſcribed C, which is the Reſult of the abſolute term divided by the third part of the Co-efficient, (that is, $\frac{RqC}{Rq} = C$;) And either of the Arches to which this Chord ſubtends be divided into three equal parts: The Chord which ſubtendeth to one of thoſe parts is an (Affirmative) Root of that Equation; which therefore hath two Affirmative Roots; ſuppoſe A, and E.

LXII. But it hath moreover a Negative Root; which is the Subtenſe of either of thoſe Arches (whoſe Chord is A, or E,) increaſed by a Trient of the whole Circumference, ſuppoſe Z. I ſay *either of thoſe Arches*; for the ſame Chord Z, which on the one ſide ſubtends a Trient increaſed by the Arch A, ſubtends on the other ſide a like Trient increaſed by E.

LXIII. But if the Equation be of this Form, $Zc - 3RqZ = RqC$: The proceſs is juſt the ſame in all Points; ſave, that then, there is but one Affirmative Root, Z; and two Negatives, A, E.

LXIV. But, in both Caſes, it muſt be ſtill obſerved, *That the Chord* C *be not greater than* 2R. (For when this happens, the Chord C, as being greater than the Diameter, cannot be inſcribed in ſuch Circle.) Or, (which is in effect the ſame;) *That the Square of Half the Abſolute term, be not greater than the Cube of a third part of the Co-efficient of the middle term.* For, the third part of that Co-efficient being Rq, and the Cube thereof R^6; and half the Abſolute quantity $\frac{1}{2}RqC$, and the Square of this $\frac{1}{4}RqqCq$; if this Square be greater than that Cube, and therefore (dividing both by Rqq) $\frac{1}{4}Cq$ greater than Rq, and (taking the Roots of both) $\frac{1}{2}C$ greater than R; then muſt C be greater than 2R the Diameter, and therefore cannot be inſcribed in the Circle. And therefore, *when this happens, ſuch Equations cannot be thus reſolved by the Triſection of an Arch.* But they may by (what are wont to be called) *Cardan's Rules*, (as I have elſewhere ſhewed,) the conſideration of which doth not belong to this place.

LXV. If an Arch to be Tripled be a Trient (or Two, Three, Four, or more Trients;) the Triple Arch will therefore be one intire Revolution, (or Two, Three, Four, or more intire Revolutions;) and the Subtenſe of the Triple will be nothing; (the beginning and end of ſuch Triple Arch being the ſame Point:) That is, $3A - \frac{Ac}{Rq}$, (or $3E - \frac{Ec}{Rq}$,) $= C\left(= \frac{Zc}{Rq} - 3Z\right) = 0$. And therefore $3A = \frac{Ac}{Rq}$, $3E = \frac{Ec}{Rq}$, $\frac{Zc}{Rq} = 3Z$. And $3Rq = Aq = Eq = Zq$. That is, (as was before ſhewed)

LXVI. *The Square of the Subtenſe of a Trient*, (or of the ſide of an Equilater inſcribed Triangle) *is equal to three Squares of the Radius.*

LXVII. If

LXVII. If an Arch to be Tripled be a Quadrant; it is manifeſt that the Subtenſe of the Triple, is equal to that of the ſingle. (For the ſame Chord, ſubtendeth, on the one ſide, to Three ſuch Arches; and, on the other ſide, but to one.) That is, $3A - \frac{Ac}{Rq} = C = A$; and therefore $2A - \frac{Ac}{Rq} = 0$; and $2Rq = Aq$. That is, *Fig.*XIV.

LXVIII. *The Square of the Subtenſe of a Quadrant* (or of the ſide of an inſcribed Quadrate) *is equal to two Squares of the Radius.*

LXIX. The ſame may be inferred, from the Biſection of a Semicircumference. For the Subtenſe of that being $2R$; and therefore (by § 9, *Chap. preced.*) $2R = B = \frac{A}{R}\sqrt{:4Rq - Aq}$: Or, (putting E for the Remainder of that Quadrant to the Semicircle) $2R = \frac{AE}{R}$: Or, (becauſe, in this caſe $E = A$,) $2R = \frac{Aq}{R}$. Therefore $2Rq = Aq$, as before; and $A = R\sqrt{2}$.

LXX. But if the Arch to be Tripled be of a Semicircle (and ſo, greater than a Trient,) the Subtenſe of the Triple will be the ſame with that of the ſingle; but with a contrary ſign, (by § 28;) and therefore (by § 36.) $\frac{Zc}{Rq} - 3Z = C = Z$; that is, $\frac{Zc}{Rq} = 4Z$, or $Zc = 4ZRq$, and $Zq = 4Rq$, and $Z = 2R$. Which is the third, or Negative Root, of the laſt mentioned Equation, $3A - \frac{Ac}{Rq} = C = A$; beſide the two Affirmatives A, E, the Subtenſes of the two Quadrants: This being equal to the Aggregate of both, with the contrary ſign.

LXXI. Moreover; becauſe the ſame Subtenſe (before mentioned,) $C = A$, ſubtends not onely to the Triple of a Quadrant on the one ſide; but alſo, on the other ſide, to the ſingle Quadrant; to a third part of this therefore the Root E is to be alſo a ſubtendent: (that is, to an Arch of 30 degrees.) That is, $3E - \frac{Ec}{R} = C = A$.

LXXII. And, becauſe (by § 15.) $Aq + AE + Eq = 3Rq$, and (by § 67.) $Aq = 2Rq$; therefore $AE + Eq = Rq$. And therefore (by reſolving this Equation) $E = \sqrt{:\frac{1}{4}Rq\ (= \frac{1}{4}Aq) + Rq: - \sqrt{\frac{1}{2}}Rq: = \sqrt{\frac{3}{2}}Rq - \sqrt{\frac{1}{2}}Rq = \frac{\sqrt{3} - \sqrt{1}}{\sqrt{2}}R\ (= \frac{\sqrt{6} - \sqrt{2}}{2}R$. And therefore, $\sqrt{2} . \sqrt{3} - 1 :: R . E$. That is,

LXXIII. *As the Subtenſe of a Quadrant, to the Subtenſe of a Trient wanting the Radius; ſo is the Radius, to the Subtenſe of the Semi-ſextant*; or, of 30 degrees: Or,

LXXIV. *As the ſide of the (inſcribed Equilater) Tetragone, to the Difference of the ſides of the Trigone and Hexagone; ſo is the Radius, to that of the Dodecagone.* (Underſtand it of the inſcribed Equilater Figures; and ſo afterward in like caſes.)

LXXV. And becauſe (as before) $E = \frac{\sqrt{3} - 1}{\sqrt{2}}R$: Therefore, $Eq = \frac{4 - 2\sqrt{3}}{2}Rq = 2Rq - Rq\sqrt{3}$, or, $2 - \sqrt{3}$ into Rq. And ſo, $1 . 2 - \sqrt{3} :: Rq . Eq$. That is,

LXXVI. *As*

Fig. XIV. LXXVI. *As the Radius, to the Exceſs of the Diameter above the Subtenſe of* 120 *degrees*, (or ſide of the inſcribed Trigone:) *ſo is the Square of the Radius, to that of the Subtenſe of* 30 *degrees*; or of the ſide of the Dodecagone. And therefore, (that of Rq to Eq, being Duplicate to that of R to E.)

LXXVII. *The Proportion of the Radius, to the Difference of the Diameter and the ſide of the inſcribed Trigone; is Duplicate to that of the Radius, to the ſide of the inſcribed Dodecagone.* And therefore,

LXXVIII. *The (Radius or) ſide of the Hexagone, and of the Dodecagone, and the Difference of the Diameter from that of the Trigone, are in continual Proportion.*

LXXIX. And (becauſe $2-\sqrt{3}$ into Rq, $=2R-R\sqrt{3}$ into R,) *The Exceſs of the Diameter above the Subtenſe of the Trient, Multiplied into the Radius; is equal to the Square of the Subtenſe of* 30 *degrees*, or the Semiſextant.

LXXX. The ſame are found by biſecting the Sextant; (for a quarter of the Trient, or half the Sextant, is the ſame;) in this manner,

LXXXI. If E be put for the Subtenſe of 30 degrees, and A for that of the Reſidue to a Semicircumference, or of 150 degrees; then becauſe the Subtenſe of a Sextant, or the double Arch of E, is R; therefore (by § 14, *Chap. preced.*) $Bq=Rq=\frac{4RqAq-Aqq}{Rq}=\frac{4RqEq-Eqq}{Rq}$: And $Rqq=4RqAq-Aqq=4RqEq-Eqq$: And (by reſolving that Equation) $2Rq\pm\sqrt{3}Rqq=2Rq\pm Rq\sqrt{3}=2\pm\sqrt{3}$ into $Rq=2R\pm R\sqrt{3}$ into R, $=Aq$, and Eq. That is,

LXXXII. *The Squares of the Subtenſes of* 150, *and of* 30 *degrees; are equal, that to the Sum, this to the Difference, (of the Diameter and ſide of the inſcribed Trigone) Multiplied by the Radius.*

LXXXIII. Again, for as much as $C=A$ ſubtends as well the Triple of the Arch A of 90 degrees, as the Triple of the Arch E of 30 degrees; therefore $Z\ (=A+E=R\sqrt{2}+\frac{\sqrt{3}-1}{\sqrt{2}}R=\frac{\sqrt{3}+1}{\sqrt{2}}R)$ is the Subtenſe of a Trient increaſed by the Arch A or E; that is, as well of degr. $210=120+90$, as of $150=120+30$. Which was before concluded at § 81. (for Z here, is the ſame with A there) and Zq (as there Aq,) $=2+\sqrt{3}$ into Rq.

LXXXIV. And the ſame is yet again found by ſubducting ($2-\sqrt{3}$ into Rq) the Square of the Subtenſe 30 degrees, out of (4Rq) the Square of the Diameter: (becauſe $150+30=180$ degrees, complete the Semicircumference:) For if from 4Rq we ſubduct $2Rq-Rq\sqrt{3}$, there remains $2Rq+Rq\sqrt{3}$, or, $2+\sqrt{3}$ into Rq, the Subtenſe of 150 degrees; and therefore alſo of 210 degrees.

CHAP.

CHAP. III.

Of the Quadruplation and Quadriſection of an ARCH *or* ANGLE.

I. IF in a Circle be inſcribed a Quadrilater, whoſe oppoſite ſides are A, A, (the ſubtenſes of a ſingle Arch) and B, D, (the ſubtenſes of the Double and Quadruple) the Diagonals will be C, C, (the ſubtenſes of the Triple) as is evident. (But it is evident alſo, that, in this Caſe, the Arch A, is leſs than a Quadrant of the whole Circumference.) *Fig.* XV.

II. And therefore (the Rect-angle of the Diagonals being equal to the two Rect-angles of the oppoſite ſides) $Cq - Aq = BD$. And therefore $\frac{Cq - Aq}{B} = D$, and $\frac{Cq - Aq}{D} = B$. That is,

III. *The Square of the Subtenſe of the Triple Arch, wanting the Square of the Subtenſe of the ſingle Arch (leſs than a Quadrant) is equal to the Rect-angle of the ſubtenſes of the Double and Quadruple.* And, divided by either of theſe, it gives the other of them.

IV. But $C + A$ into $C - A$ is equal to $Cq - Aq$. And therefore, $B . C + A :: C - A . D$. That is,

V. *As the Subtenſe of the double Arch, is to the ſum of the Subtenſes of the Triple, and of the ſingle (this being leſs than a Quadrant;) ſo is the exceſs of the Subtenſe of the Triple above that of the ſingle, to that of the Quadruple.*

VI. And becauſe (by § 8, *Chap. preced.*) $C = 3A - \frac{Ac}{Rq}$; and therefore $Cq = 9Aq - \frac{6Aqq}{Rq} + \frac{Acc}{Rqq}$: Therefore $Cq - Aq = 8Aq - \frac{6Aqq}{Rq} + \frac{Acc}{Rqq} = \frac{8RqqAq - 6RqAqq + Acc}{Rqq} = BD$.

VII. But (by § 7 and 9, *Chap.* 29.) $B = \frac{A}{R}\sqrt{:4Rq - Aq}$. And therefore $\frac{Cq - Aq}{B} = \frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$. (For if $Cq - Aq = \frac{8RqqAq - 6RqAqq + Acc}{Rqq}$ be divided by $\frac{A}{R}$, it is $\frac{8RqqA - 6RqAc + Aqc}{Rc}$: And this again divided by $4Rq - Aq$, is $\frac{2RqA - Ac}{Rc}$: But this laſt Diviſion being by $4Rq - Aq$, whereas (according to the value of B) it ſhould be divided only by the Square Root hereof, therefore we are to reſtore a Multiplication by that Root; which makes it $\frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$.)

VIII. And then, turning the Equation into an Analogy, $Rc . 2RqA - Ac :: \sqrt{:4Rq - Aq} . D$. That is,

Fig. XV. IX. *As the Cube of the Radius, to the Subtenſe of the ſingle Arch (leſs than a Quadrant) Multiplied into the double Square of the Radius wanting the Square of the Subtenſe, ſo is the Subtenſe of what this Arch wants of a Semicircumference, to the Subtenſe of the Quadruple Arch.*

X. Or thus; (dividing the two firſt terms by Rq,) $R \cdot 2A - \frac{Ac}{Rq} :: \surd : 4Rq - Aq \cdot D$. That is,

XI. *As the Radius, to the double of the Subtenſe of the ſingle Arch (leſs than a Quadrant) wanting the Cube of that Subtenſe divided by the Square of the Radius: So is the Subtenſe of what that ſingle Arch wants of the Semicircumference, to the Subtenſe of the Quadruple Arch.*

XII. But (by § 8, *Chap. preced.*) $2A - \frac{Ac}{Rq} = C - A$. And therefore,

XIII. *As the Radius to the exceſs of the Triple Arch above that of the ſingle (leſs than a Quadrant:) ſo is the Subtenſe of what that ſingle Arch wants of a Semicircumference, to the Subtenſe of the Quadruple Arch.*

XIV. The ſame may be thus alſo demonſtrated: Becauſe (by § 9, *Chap.* 29.) $\frac{A}{R}\surd : 4Rq - Aq : = \frac{A\surd : 4Rq - Aq}{R} : = B$, is the Subtenſe of the double Arch of A: Therefore (by the ſame reaſon) $\frac{B}{R}\surd : 4Rq - Bq : = D$, the Subtenſe of the double Arch of B; that is, of the Quadruple Arch of A.

XV. And, becauſe $Bq = 4Aq - \frac{Aqq}{Rq}$: Therefore, for $4Rq - Bq$, we may put $4Rq - 4Aq + \frac{Aqq}{Rq}$, or $\frac{4Rqq - 4RqAq + Aqq}{Rq}$: And the Quadratick Root hereof, $\frac{2Rq - Aq}{R}$ $(= \surd : 4Rq - Bq :)$ Multiplied into $\frac{A\surd : 4Rq - Aq}{Rq}$ $\left(= \frac{B}{R}\right)$ makes $\frac{2RqA - Ac}{Rc}\surd : 4Rq - Aq : = \left(\frac{B}{R}\surd : 4Rq - Bq : =\right) D$: As before,

Fig. XVI. XVI. We may alſo, to the ſame purpoſe, (and with the ſame event,) inſcribe a Quadrilater, ſo as that A, D, and A, B, may be oppoſite ſides, and B, C, Diagonals. For then $BC - BA = DA$. And therefore $2A - \frac{Ac}{Rq}$ $(= C - A)$ into $(B =)$ $\frac{A}{R}\surd : 4Rq - Aq :$ will equal DA. That is, $\frac{2RqAq - Aqq}{Rc}\surd : 4Rq - Aq : = DA$. And $\frac{2RqA - Ac}{Rc}\surd : 4Rq - Aq : = D$; as before. But of this we ſhall ſay more at § 86. *&c.*

Fig. XV. XVII. But, for as much as the ſame D, Subtends not only the Quadruple of the Arch A, but alſo the Quadruple of the Arch E, (which therefore, together with the Arch A, will complete a Quadrant of the whole Circumference:) it may in like manner be ſhewed, that $\frac{2RqE - Ec}{Rc}\surd : 4Rq - Eq : = D = \frac{2RqA - Ac}{Rc}\surd : 4Rq - Aq :$ And therefore,

XVIII. *An Arch leſs than a Quadrant, and the Arch which this wants of a Quadrant, have both the ſame Subtenſe of the Quadruple Arch,* D. And, accordingly, AE, are two Affirmative Roots of that Equation.

XIX. But

XIX. But there are yet two other Roots (but both Negative, as will after appear) of the ſame Equation, (which we will call P, S;) whereof one ſubtends a Quadrant increaſed by the Arch A; the other, a Quadrant increaſed by the Arch E. For it is manifeſt (by what is ſaid at § 23, *Chap. preced.*) that theſe alſo muſt have the ſame Subtenſe of the Quadruple Arch, with A and E. For Four times $\frac{1}{4}+A$, is $1+4A$, and will therefore have the ſame Subtenſe with 4A. And the like of Four times $\frac{1}{4}+E$, which is $1+4E$, whoſe Subtenſe is the ſame with that of 4 E. (And the like will follow, in caſe two, three, or more Quadrants be thus increaſed.) And conſequently, *Fig.* XVII.

XX. *An Arch greater than a Quadrant*, (or than two, three, or more Quadrants) *will require the ſame Subtenſe of its Quadruple Arch, with its exceſs above a Quadrant*, (or above theſe two, three, or more Quadrants.)

XXI. But the ſame P, ſubtends as well to a Quadrant increaſed by the Arch of A, as to three Quadrants wanting the ſaid Arch; as alſo to a Semicircumference (or two Quadrants) increaſed by the Arch of E, or wanting that Arch. (As is manifeſt to view by the Scheme.) And, in like manner, S ſubtends as well to a Quadrant increaſed by the Arch of E, as to three Quadrants wanting that Arch; as alſo to a Semicircumference (or two Quadrants) increaſed by the Arch of A, or wanting that Arch.

XXII. Now, that P, S, are Negative Roots, will thus appear. For, ſuppoſing (for inſtance) $\frac{2RqP-Pc}{Rc}\sqrt{:4Rq-Pq:=D}$; and P, a ſubtendent of an Arch greater than a Quadrant: (But, leſs than three Quadrants; otherwiſe it is the ſame as if it were leſs than one Quadrant: For the ſame Chord which ſubtends an Arch greater than three Quadrants, ſubtends alſo to leſs than one:) P will in this caſe be greater than $\sqrt{2Rq}$ the Subtenſe of a Quadrant, and therefore $2Rq-Pq$ a Negative quantity (becauſe of greater quantity ſubtracted from a leſſer;) and therefore alſo P muſt be Negative, that ſo $2RqP-Pc$ (compounded by the Multiplication of two Negatives) may be a Poſitive quantity, and therefore the whole $\frac{2RqP-Pc}{Rc}\sqrt{:4Rq-Pq:=D}$ Affirmative alſo. (And what is ſaid of P, holds in like manner of S.)

XXIII. But if we chuſe to make P Affirmative, then muſt $\frac{2RqP-Pc}{Rc}\sqrt{:4Rq-Aq:=-D}$, be Negative: And therefore (changing the ſigns) $\frac{Pc-2RqP}{Rc}\sqrt{:4Rq-Aq:=D}$, Affirmative. (And the like of S.) But, of this, more afterwards.

XXIV. But for what reaſon, the Equation $\frac{2RqA-Ac}{Rc}\sqrt{:4Rq-Aq:=D}$, or $\frac{2RqE-Ec}{Rc}\sqrt{:4Rq-Eq:=D}$, hath two Affirmative Roots A, E; and two Negatives P, S; for the ſame reaſon will the Equation $\frac{Pc-2RqP}{Rc}\sqrt{:4Rq-Pq:=D}$, or $\frac{Sc-2RqS}{Rc}\sqrt{:4Rq-Sq:=D}$, have two Negatives A, E; and P, S, Affirmatives.

XXV. If now we conſider the Quadrilater, whoſe ſaid oppoſite ſides are A, A, and E, P; then (becauſe the Arches A, E, do together make up a Quadrant) the Diagonals Q, Q, are ſubtenſes of a Quadrant, (or ſides of an inſcribed Square) and therefore (by § 68, *Chap. preced.*) $Qq=2Rq$, and $Q=\sqrt{:2Rq}=R\sqrt{2}$.

Fig. XVII.

XXVI. And therefore $Qq - Aq = 2Rq - Aq = EP$; and consequently $\frac{2Rq-Aq}{P} = E$, and $\frac{2Rq-Aq}{E} = P$.

XXVII. But the ſame P doth alſo ſubtend a Semicircumference wanting the Arch of E: And therefore $\surd : 4Rq - Eq : = P = \frac{2Rq-Aq}{E}$: And $\surd : 4Rq - Pq : = E = \frac{2Rq-Aq}{P}$.

XXVIII. And (by the ſame reaſon) taking a Quadrilater whoſe oppoſite ſides are E, E, and A, S; we have the Diagonals $Q = \surd 2Rq$: And $Qq - Eq = 2Rq - Eq = AS$. And conſequently (becauſe the Arches A and S do complete the Semicircumference) $\surd : 4Rq - Aq : = S = \frac{2Rq-Eq}{A}$; and $\surd : 4Rq - Sq : = A = \frac{2Rq-Eq}{S}$.

XXIX. Now becauſe (as at § 27.) $\surd : 4Rq - Eq : = P = \frac{2Rq-Aq}{E}$; therefore $4Rq - Eq = Pq = \frac{4Rqq - 4RqAq + Aqq}{Eq}$. And therefore $4RqEq - Eqq = PqEq = 4Rqq - 4RqAq + Aqq$; and $Aqq + Eqq = 4AqRq + 4EqRq - 4Rqq$. (And, in like manner, becauſe $\surd : 4Rq - Aq : = S = \frac{2Rq-Eq}{A}$; therefore $4Rq - Aq = Sq = \frac{4Rqq - 4RqEq + Eqq}{Aq}$; and $4RqAq - Aqq = SqAq = Rqq = 4Rqq - 4RqEq + Eqq$; and $4AqRq + 4EqRq - 4Rqq = Aqq + Eqq$.)

XXX. Now the Legs A, E, contain a Seſquiquadrantal Angle, or of 135 Degrees: (As being an Angle in the Peripherie, ſtanding on an Arch of three Quadrants:) And therefore,

XXXI. *In a Right-lined Triangle, whoſe Angle at the Top is* 135 *Degrees, if the double of the Aggregate of the Squares of the Legs containing it,* ($2Aq + 2Eq$,) *wanting the Square of the Baſe* ($Qq = 2Rq$,) *be Multiplied into the Square of the Baſe* ($2Rq$;) *the Product* ($4AqRq + 4EqRq - 4Rqq = 2Aq + 2Eq - 2Rq$ into $2Rq$,) *is equal to the Biquadrates of the Legs* ($Aqq + Eqq$.) So that,

XXXII. From hence appears, *A convenient Method for Adding of Biquadrates.*

XXXIII. *The Subduction of Biquadrates,* may (with a little alteration) be performed almoſt in the ſame manner. But it is more conveniently done by *Multiplying the Sum of the Squares, by the Difference of them.* (For $Aq + Eq$ into $Aq - Eq$ is equal to $Aqq - Eqq$.) But that is a ſpeculation not of this place.

XXXIV. Again, *In ſuch Triangle (whoſe Angle at the Top is of* 135 *Degrees) If the double of the Aggregate of the Squares of the Legs, be Multiplied into the Square of the Baſe; the Product is equal to the Aggregate of the Biquadrates of all the ſides.* For, ſince $4AqRq + 4EqRq - (Qqq =) 4Rqq = Aqq + Eqq$; therefore $Aqq + Eqq + Qqq = 4AqRq + 4EqRq = 2Aq + 2Eq$ into $(2Rq =) Qq$.

XXXV. Again,

XXXV. Again, becauſe (as at § 27, 28.) $\frac{2Rq-Aq}{P}=E=\sqrt{:4Rq-Pq}$: and therefore $\frac{4Rqq-4RqAq+Aqq}{Pq}=Eq=4Rq-Pq$, and $4Rqq-4RqAq+Aqq=PqEq=4PqRq-Pqq$. Therefore $Pqq+Aqq=4PqRq+4AqRq-4Rqq$. (And in like manner, becauſe $\frac{2Rq-Eq}{S}=A=\sqrt{:4Rq-Sq}$; and therefore $\frac{4Rqq-4EqRq+Eqq}{Sq}=Aq=4Rq-Sq$, and $4Rqq-4EqRq+Eqq=SqAq=4SqRq-Sqq$: Therefore, $Sqq+Eqq=+SqRq+4EqRq-4Rqq$.) *Fig.* XVII.

XXXVI. But both A, P, and alſo E, S, contain a Semiquadrantal Angle, or of 45 Degrees: (As being an Angle in the Periphery ſtanding on a Quadrantal Arch;) And one of the Angles at the Baſe, Obtuſe. And therefore,

XXXVII. *In a Right-lined Triangle, whoſe Angle at the Top is of* 45 *Degrees, or half a Right-angle (one of the other being Obtuſe) If the double of the Aggregate of the Squares of the Legs* (as $2Pq+2Aq$) *wanting the Square of the Baſe* ($Qq=2Rq$) *be Multiplied into the Square of the Baſe* ($2Rq$) *the Product* ($4PqRq+4AqRq-4Rqq=2Pq+2Aq-2Rq$ into $2Rq$,) *is equal to the Biquadrates of the Legs*, ($Pqq+Aqq$,) In like manner, $2Sq+2Eq-2Rq$ into $2Rq$, $=4SqRq+4EqRq-4Rqq=Sqq+Eqq$. So that

XXXVIII. Here is *another Method of Adding Biquadrates.*

XXXIX. And likewiſe; *In ſuch Triangle, whoſe Angle at the Top is of* 45 *Degrees, (and one of the other Obtuſe,) if the double of the Aggregate of the Squares of the Legs, be Multiplied into the Square of the Baſe; the Product is equal to the Biquadrates of all the ſides.* For, becauſe $4PqRq+4AqRq-4Rqq=Pqq+Aqq$; therefore $Pqq+Aqq+(4Rqq=)Qqq=4PqRq+4AqRq=2Pq+2Aq$ into $(2Rq=)Qq$. And, becauſe $4SqRq+4EqRq-4Rqq=Sqq+Eqq$, therefore $Sqq+Eqq+(4Rqq=)Qqq=4SqRq+4EqRq=2Sq+2Eq$ into $(2Rq=)Qq$.

XL. Furthermore; If in a Circle be inſcribed a Quadrilater, whoſe oppoſite ſides are S, A, and Q, Q, and the Diagonals P, P, (as in the Scheme;) Then *Fig.* XVIII. $Pq-(Qq=)2Rq=SA$. And therefore, $\frac{Pp-2Rq}{A}=S=\sqrt{:4Rq-Aq}$: And $\frac{Pq-2Rq}{S}=A=\sqrt{:4Rq-Sq}$: And therefore, $\frac{Pqq-4PqRq+4Rqq}{Aq}=Sq=4Rq-Aq$, and $\frac{Pqq-4PqRq+4Rqq}{Sq}=Aq=4Rq-Sq$. And conſequently $Pqq-4PqRq+4Rqq=AqSq=4RqAq-Aqq=4RqSq-Sqq$.

XLI. And, by the ſame reaſon, If a Quadrilater be inſcribed whoſe oppoſite ſides are E, P, and Q, Q; and the Diagonals S, S: Then $Sq-(Qq=)2Rq=EP$: And therefore, $\frac{Sq-2Rq}{E}=P=\sqrt{:4Rq-Eq}$: And $\frac{Sq-2Rq}{P}=E=\sqrt{:4Rq-Pq}$: And therefore, $\frac{Sqq-4SqRq+4Rqq}{Pq}=Eq=4Rq-Pq$: And $\frac{Sqq-4SqRq+4Rqq}{Eq}=Pq=4Rq-Eq$. And conſequently, $Sqq-4SqRq+4Rqq=PqEq=4PqRq-Pqq=4EqRq-Eqq$.

XLII. And either way, we may conclude, $Sqq+Pqq=4SqRq+4PqRq-4Rqq$.

XLIII. But

Fig. XVIII. XLIII. But P, S, contain half a Right-angle, or Angle of 45 Degrees; (as being an Angle in the Periphery ſtanding on a Quadrantal Arch;) and both the other Angles Acute. And therefore,

XLIV. *In a Right-lined Triangle, whoſe Angle at the Top is 45 Degrees, or half a Right Angle: (and both at the Baſe, acute:) If the double of the Aggregate of the Squares of the Legs* (as $2Sq+2Pq$) *wanting the Square of the Baſe* ($Qq=2Rq$) *be Multiplied into the Square of the Baſe* ($2Rq$) *the Product* ($4SqRq+4PqRq-4qq=2Sq+2Pq-2Rq$ into $2Rq$) *is equal to the Biquadrates of the Legs.* ($Sqq+Pqq$.)

XLV. And this is *a third Method of Adding Biquadrates.*

XLVI. *And likewiſe, in ſuch Triangles, (whoſe Angle at the Top is of 45 Degrees, and both the others acute,) if the double of the Aggregate of the Squares of the Legs, be Multiplied into the Square of the Baſe; the Product is equal to the Biquadrates of all the ſides.* For, ſince $Sqq+Pqq=4SqRq+4PqRq-4Rqq$, therefore $Sqq+Pqq+$ ($4Rqq=$) $Qqq=4SqRq+4PqRq=2Sq+2Pq$ into $2Rq$.

XLVII. Theſe Theorems thus demonſtrated ſeverally; whether the Angle at the Top be of 135 Degrees, or of 45 Degrees; and this whether the Triangle be Acute-angled, or Obtuſe-angled, (to either of which we may refer the Rect-angled;) may be thus reduced to theſe Generals.

XLVIII. *In a Right-lined Triangle, whoſe Angle at the Vertex is either of 135 Degrees, or of 45 Degrees; the double Aggregate of the Squares of the Legs containing it, wanting the Square of the Baſe, Multiplied into the Square of the Baſe, is equal to the Biquadrates of the two Legs.* (Which is the Addition of Biquadrates.) By § 31, 37, 44.

XLIX. *And that double Aggregate of the Squares of the Legs, Multiplied into the Square of the Baſe, is equal to the Biquadrates of all the three ſides.* By § 34, 39, 46.

L. Now, the Equation, (at § 29.) $4AqRq+4EqRq-4Rqq=Aqq+Eqq$ (and the other like to it at § 35, 42.) is a Quadratick Equation of a Plain Root: Whereof the Root is $2Rq$; the Co-efficient of the middle Term, $2Aq+2Eq$; which is therefore equal to the ſum of two Quantities, whoſe Rect-angle is equal to the Abſolute Quantity $Aqq+Eqq$.

LI. If we therefore order this according to the Rule of other Equations of the ſame form; and, accordingly, from $Aqq+2AqEq+Eqq$ (the Square of half the Co-efficient $Aq+Eq$) we ſubtract (the Abſolute Quantity) $Aqq+Eqq$; the Remainder is $2AqEq$: And the Square Root of this ($\sqrt{2AqEq}=AE\sqrt{2}$) Added to, or Subducted from, half the Co-efficient $Aq+Eq$, gives the Root of that Equation $Aq+Eq\pm AE\sqrt{2}=2Rq=Qq$.

LII. But, of this Ambiguous Equation, 'tis evident that we are to make choiſe of the greater Root, in the caſe of § 29: Becauſe the Angle at the Vertex (135 Degrees) is greater than a Right-angle; and therefore the Square of the Baſe (Qq) is to be greater than ($Aq+Eq$) the Squares of the two ſides containing it. And therefore $Aq+Eq+AE\sqrt{2}=2Rq=Qq$. That is,

LIII. *If to the Squares of the Legs containing an Angle of 135 Degrees, (or three halfs of a Right-angle,) we add the Rect-angle of thoſe Legs Multiplied by* $\sqrt{2}$; *the Aggregate is equal to the Square of the Baſe.*

LIV. in the ſame manner may be ſhewed, that the Equations of § 35. $Pqq + Aqq = 4PqRq + 4AqRq - 4Rqq$, (or $Sqq + Eqq = 4SqRq + 4EqRq - 4Rqq$,) and of § 42. $Sqq + Pqq = 4SqRq + 4PqRq - 4Rqq$; are Quadratic Equations of a Plain Root $2Rq$. But, in all theſe ('tis manifeſt) the leſſer Root is to be choſen, becauſe the Angle at the Vertex (being of 45 Degrees) is leſs than a Right Angle; and therefore the Square of the Baſe leſs than the two Squares of the Legs. And therefore, the Root, $Pq + Aq - PA\sqrt{2} = 2Rq$; and $Sq + Eq - SE\sqrt{2} = 2Rq$; and $Sq + Pq - SP\sqrt{2} = 2Rq = Qq$. That is, *Fig.* XVIII.

LV. *If from the Squares of the Legs containing an Angle of* 45 *Degrees* (or half a Right-angle,) *we ſubtract the Rect-angle of theſe Legs Multiplied by* $\sqrt{2}$: *the Remainder is equal to the Square of the Baſe.*

LVI. Or we may put both together, thus: *If to the Squares of the Legs, be Added, if they contain an Angle of* 135 *Degrees; or ſubtracted thence, if they contain an Angle of* 45 *Degrees; a Rect-angle of thoſe Legs Multiplied into* $\sqrt{2}$: *The Reſult is equal to the Square of the Baſe.* By § 53, 55.

LVII. We are next to Note, That the ſubtenſes E and P, as alſo A and S, (whoſe two Arches do together make up a Semicircumference,) do (by § 9, *Chap.* 29.) require the ſame Subtenſe of the double Arch: And therefore much more, the ſame Subtenſe of the Quadruple. That is, $\frac{EP}{R}$ is the Subtenſe of the double Arch both of E, and of P: And $\frac{AS}{R}$, of the double Arch of A, and of S. *Fig.* XIX

LVIII. The Subtenſe therefore of the Triple Arch of E, (leſs than a Quadrant, and therefore, much more, leſs than a Trient,) is $\frac{PqE - RqE}{Rq}$, (by § 2, 5, *Chap. preced.*) as being the Square of the Subtenſe of the double Arch $\frac{EqPq}{Rq}$, wanting the Square of the Subtenſe of the ſingle Arch Eq, divided by the Subtenſe of the ſingle Arch E.

LIX. But the ſame Subtenſe of that Triple Arch, (by § 8, 9, *Chap. preced.*) is $\frac{3RqE - Ec}{Rq}$.

LX. Therefore $\frac{PqE - RqE}{Rq} \left(= \frac{PqE}{Rq} - E\right) = \frac{3RqE - Ec}{Rq} \left(= 3E - \frac{Ec}{Rq}.\right)$ And $\frac{PqE}{Rq} = 4E - \frac{Ec}{Rq}$, the Aggregate of the ſubtenſes of the Triple and ſingle.

LXI. Which may alſo be thus proved: Becauſe $Pq + Eq = 4Rq$ (as being in a Semicircle,) and therefore $Pq = 4Rq - Eq$, and $PqE = 4RqE - Ec$, or $PqE - RqE = 3RqE - Ec$. Therefore is $\frac{PqE - RqE}{Rq} = \frac{3RqE - Ec}{Rq}$ the Subtenſe of the Trible; and $\frac{PqE}{Rq} = \frac{4RqE - Ec}{Rq}$ the Aggregate of the ſubtenſes of the Triple and ſingle.

LXII. And, by juſt the ſame reaſon, $\frac{SqA - RqA}{Rq} \left(= \frac{SqA}{Rq} - A\right) = \frac{3RqA - Ac}{Rq}$ $\left(= 3A - \frac{Ac}{Rq}\right)$ is the Subtenſe of the Triple Arch of A: And $\frac{SqA}{Rq} = 4A - \frac{Ac}{Rq}$ the Aggregate of the ſubtenſes of the Triple and ſingle.

LXIII. Now

Fig. XIX LXIII. Now the Arch of P, (a Quadrant increaſed by A, its greater Segment) being greater than a Trient, but leſs than two Trients; the Subtenſe of its Triple Arch is $\frac{Rq\,P - Eq\,P}{Rq}$ (by § 32. *Chap. preced.*) And $\frac{Pc - 3\,Rq\,P}{Rq}$. (By § 37. *Chap. preced.*)

LXIV. And therefore, $\frac{Rq\,P - Eq\,P}{Rq} \left(= P - \frac{Eq\,P}{Rq}\right) = \frac{Pc - 3\,Rq\,P}{Rq} \left(= \frac{Pc}{Rq} - 3\,P\right)$ the Subtenſe of the Triple Arch; and $4\,P - \frac{Pc}{Rq} = \frac{Eq\,P}{Rq}$, the Difference of the Subtenſes of the ſingle and Triple; that is, the Exceſs of the Subtenſe of the ſingle above that of the Triple.

LXV. But the Arch S (a Quadrant increaſed by its leſſer Segment E) becauſe it may be either leſſer or greater than a Trient, according as the Arch E is leſs or greater than 30 Degrees; the Subtenſe of the Triple Arch will be either $\frac{Aq\,S}{Rq} - S = 3\,S - \frac{Sc}{Rq}$, if the Arch S be leſs than a Trient; or, if greater, $S - \frac{Aq\,S}{Rq} = \frac{Sc}{Rq} - 3\,S$. And, accordingly, $\frac{Aq\,S}{Rq} = 4\,S - \frac{Sc}{Rq}$, will be either the Sum or Difference of the ſubtenſes of the ſingle and Triple Arch, according as S is leſs or greater than a Trient.

LXVI. Moreover; having ſhewed (at § 2.) that in a Quadruplication of an Arch leſs than a Quadrant, $Cq - Aq = BD$, (as wherein the Subtenſe of the Triple is greater than that of the ſingle; and therefore C, C, Diagonals, and A, A, oppoſite ſides:) Now, if the Arch to be Quadrupled be greater than a Quadrant, (but leſs than three Quadrants) as that of P or S; the Subtenſe of the ſingle will be greater than that of the Triple. For, ſuppoſing the ſingle Arch to be $\frac{1}{2} \mp A$ (and A leſs than $\frac{1}{4}$) the Triple will be $\frac{1}{2} \mp 3\,A$; the Subtenſe of which will be the ſame with that of $\frac{1}{2} \mp 3\,A$ (for one whole revolution is, in this caſe, Equivalent to nothing;) and this (ſo long as A remains leſs than $\frac{1}{4}$) will be farther (either in exceſs or defect) from a Semicircumference (and therefore require a leſs Chord,) than $\frac{1}{2} \mp A$.

LXVII. And therefore, in this caſe, P, P, (or S, S,) become Diagonals, and C, C, oppoſite ſides. And, conſequently, $Pq - Cq = BD$ (and $Sq - Cq = bD$;) And $\frac{Pq - Cq}{B} = D = \frac{Sq - cq}{b}$. That is,

LXVIII. *The Square of the Subtenſe of an Arch greater than a Quadrant (but leſs than three Quadrants) wanting the Square of the Subtenſe of the Triple Arch; is equal to the Rect-angle of the ſubtenſes of the Double and Quadruple.* And therefore, divided by one of theſe, it gives the other.

LXIX. But $P + C$ into $P - C$, is equal to $Pq - Cq$. And therefore, $B \,.\, P + C :: P - C \,.\, D$. (And, in like manner, $b \,.\, S + c :: S - c \,.\, D$.) That is,

LXX. *As the Subtenſe of the double Arch, to the Aggregate of the ſubtenſes of the Triple, and of the ſingle (greater than a Quadrant, but leſs than three Quadrants;) So is the exceſs of the Subtenſe of the ſingle Arch above that of the Triple, to the Subtenſe of the Quadruple.*

LXXI. Since

LXXI. Since therefore the Subtenſe of the Triple Arch $P = \frac{1}{3} + A$ (being greater than a Trient) is $C = \frac{Pc}{Rq} - 3P$; whoſe Square is $\frac{Pcc}{Rq} - \frac{6Pqq}{Rq} + 9Pq$: If this be taken from Pq, and the Remainder ($Pq - Cq = -\frac{Pcc}{Rqq} + \frac{6Pqq}{Rq} - 8Pq$,) divided by $B = \frac{P}{R}\sqrt{:4Rq - Pq:}$ (as at § 7;) the Reſult is $\frac{Pq - Cq}{B} = \frac{Pc - 2RqP}{Rc}\sqrt{:4Rq - Pq:} = D$. That is, dividing it firſt by $\frac{P}{R}$, and the Reſult by $4Rq - Pq$, or by $-Pq + 4Rq$, and then reſtoring a Multiplication by $\sqrt{:4Rq - Pq.}$) *Fig.* XIX

LXXII. And therefore (changing the equality in an Anology) $Rc . Pc - 2RqP :: \sqrt{:4Rq - Pq} . D$.

LXXIII. The ſame will happen if we take the Arch $S = \frac{1}{3} + E$. For though this may be either greater or leſs than a Trient, according as E is greater or leſs than 30 Degrees; and (accordingly) the Triple thereof, either $\frac{Sc}{Rq} - 3S$, or $3S - \frac{Sc}{Rq}$: Yet this doth not alter the caſe at all; for, either way, the Square of it is the ſame. And therefore (making the Subduction and Diviſion, as § 71.) $\frac{Sc - 2RqS}{Rc}\sqrt{:4Rq - Sq:} = D$. And $Rc . Sc - 2RqS :: \sqrt{:4Rq - Sq} . D$. That is,

LXXIV. *As the Cube of the Radius, to the Subtenſe of an Arch greater than a Quadrant (but leſs than three Quadrants,) Multiplied into the Square of the Subtenſe, wanting two Squares of the Radius; ſo is the Subtenſe of its Difference from a Semicircumference, to the Subtenſe of the Quadruple Arch.*

LXXV. Or thus; $R . \frac{Pc}{Rq} - 2P :: E . D$: Or, $R . \frac{Sc}{Rq} - 2S :: A . D$. That is,

LXXVI. *As the Radius, to the Cube of the Subtenſe of an Arch (greater than a Quadrant, but leſs than three Quadrants) divided by the Square of the Radius, wanting the double of that Subtenſe; ſo is the Subtenſe of the Difference from a Semicircumference, to the Subtenſe of the Quadruple Arch.*

LXXVII. Or thus, becauſe $\frac{Pc}{Rq} - 2P = C + P$; And alſo (in caſe the Arch S be alſo greater than a Trient) $\frac{Sc}{Rq} - 2S = \mathfrak{c} + S$. Therefore, $R . P + C :: A . D$. (And $R . P + \mathfrak{c} :: E . D$.) That is,

LXXVIII. *As the Radius, to the Aggregate of the Subtenſes of the Triple Arch and of the ſingle (this being greater than a Trient, but leſs than two Trients,) ſo is the Subtenſe of its Difference from a Semicircumference, to the Subtenſe of the Quadruple Arch.*

LXXIX. But if the Arch S (though greater than a Quadrant) be leſs than a Trient; or greater than two Trients, but leſs than three Quadrants: That is, $\frac{Sc}{Rq} - 2S = S - \mathfrak{c}$. And therefore, $R . S - \mathfrak{c} :: A . D$. That is,

LXXX. *As the Radius, to the Subtenſe of an Arch greater than a Quadrant, but leſs than a Trient (or greater than two Trients, but leſs than three Quadrants) wanting the Subtenſe of the Triple Arch; ſo is the Subtenſe of its Difference from a Semicircumference, to the Subtenſe of the Quadruple Arch.*

Fig. XIX. LXXXI. All which are evident from the Scheme; where the Chord D ſubtends the Quadruple of the Arches of A, E, P, and S: And B ſubtends the double of the Arches E and P, and **b** the double of the Arches of A and S.

LXXXII. And, in the Quadrilater whoſe ſides B, D, be oppoſite and Parallel; and C, C, oppoſite ſides; and P, P, Diagonals; $Pq - Cq = BD$, and $\frac{Pq - Cq}{B} = D$. And likewiſe, in the Quadrilater wherein **b** D are oppoſite and Parallel; **c c** oppoſite ſides; and S S Diagonals; $Sq - \mathbf{c}q = \mathbf{b}D$, and $\frac{Sq - \mathbf{c}q}{\mathbf{b}} = D$.

LXXXIII. And, in the ſame Figure, where not only the Arch of P, but of S alſo, are ſuppoſed greater than a Trient; two of the Chords S, S, (as well as P, P,) cut the Chord D.

Fig. XX. LXXXIV. But in the other Figure, where the Arch of S is ſuppoſed (greater than a Quadrant, but) leſs than a Trient; the caſe is ſomewhat different. For here **b** (the Subtenſe of the double Arch of S) falling on the other ſide of D (the Subtenſe of the Quadruple,) the Chord D is not cut by any of the Chords S.

LXXXV. But it comes to the ſame paſs, for theſe two Chords S S (whether they cut or not cut the Chord D,) being no ingredients of the inſcribed Quadrilater, (but ſerve only to ſhew that **b** is the Subtenſe of the double Arch;) it is however, $Sq - \mathbf{c}q = \mathbf{b}D$.

Fig. XVI LXXXVI. The ſame things as before, may be yet otherwiſe demonſtrated (and more commodiouſly) in this manner; Namely, if inſtead of the Quadrilater whoſe four ſides and two Diogonals are A, A, C, C, B, D; we take A, A, B, B, C, D; (taking the ſubtences of the ſingle and double, twice; but, of the Triple, and Quadruple, once:) with almoſt the ſame variety of caſes, as before. For,

LXXXVII. If the Subtenſe of the ſingle Arch be A (or E,) leſs than a Quadrant; then A, B, and A, D, will be oppoſite ſides; and B, C, Diagonals. And therefore, $CB - AB = AD$. And conſequently $2A - \frac{Ac}{Rq}$ $(= C - A)$ into $(B =)$ $\frac{A}{R}\sqrt{:4Rq - Aq:}$ equal to AD. That is, $\frac{2RqAq - Aqq}{Rc}\sqrt{:4Rq - Aq:} = AD$. And $\frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$, as before. And for the ſame reaſon, $\mathbf{c}\mathbf{b} - E\mathbf{b} = ED = \frac{2RqEq - Eqq}{Rc}\sqrt{:4Rq - Eq}$. And $\frac{2RqE - Ec}{Rc}\sqrt{:4Rq - Eq:} = D$.

Fig. XXI LXXXVIII. If the Subtenſe of the ſingle Arch be P (or S) greater than a Quadrant, and even greater than a Trient: (but leſs than two Trients:) Then B, C, and B, P, (or B, S,) will be oppoſite ſides; and D, P, (or D, S,) Diagonals. And therefore $BC + BP = PD$, (or $BC + BS = SD$.) And conſequently, $\frac{Pc}{Rq} - 2P$ $(= C + A)$ into $(B =)$ $\frac{P}{R}\sqrt{:4Rq - Pq:}$ equal to PD. That is, $\frac{Pqq - 2RqPq}{Rc}\sqrt{:4Rq - Pq:} = PD$. And $\frac{Pc - 2RqP}{Rc}\sqrt{:4Rq - Pq:} = D$: As before. And, by the ſame reaſon, $BC + BS = SD$ (if S alſo be greater than a Trient) and $\frac{Sc - 2RqS}{Rc}\sqrt{:4Rq - Sq:} = D$.

LXXXIX. But

LXXXIX. But if the ſingle Arch be that of S (greater than a Quadrant, but) leſs than a Trient; (or P greater than two Trients, but leſs than three Quadrants;) then B, C, and D, S, are oppoſite ſides; and B, S, Diagonals. And therefore, $BS - BC = DS$. And conſequently, $2S - \frac{Sc}{Rq}$ $(= S - C)$ into $(B=) \frac{S}{R} \sqrt{:} 4Rq - Sq :=SD$. That is, $\frac{2RqSq - Sqq}{Rc} \sqrt{:} 4Rq - Sq :=SD$. And $\frac{2RqS - Sc}{Rc} \sqrt{:} 4Rq - Sq :=D$: As before. And in like manner, $BP - BC = PD$ (if the Arch of P be greater than two Trients, which is the ſame as if leſs than one;) and $\frac{2RqP - Pc}{Rc} \sqrt{:} 4Rq - Pq :=D$. *Fig.* XXII.

XC. From all which ariſeth this General Theorem: *The Rect-angle of the Subtenſes of the ſingle and of the Quadruple Arch, is equal to the Subtenſe of the double Multiplied into the Exceſs of the Subtenſe of the Triple above that of the ſingle, in caſe this be leſs than a Quadrant (or more than three Quadrants;) or, into the Exceſs of the Subtenſe of the ſingle above that of the Triple, in caſe the ſingle be more than a Quadrant but leſs than a Trient (or more than two Trients, but leſs than three Quadrants;)* or, laſtly, into the Sum of the Subtenſes of the Triple and *ſingle, in caſe this be more than a Trient, but leſs than two Trients.* That is, $AD :=B$ into

$C - A$; if the Arch of A be leſs than a Quadrant, or greater than three Quadrants.

$A - C$; if it be greater than a Quadrant, but leſs than a Trient; or greater than two Trients, but leſs than three Quadrants.

$A + C$; if it be greater than a Trient, but leſs than two Trients.

XCI. And, univerſally, $\frac{2RqA \backsim Ac}{Rc} \sqrt{:} 4Rq - Aq :=D$. That is, *if the Difference of* $2Rq A$ *and* Ac (whereof that is the greater if the ſingle Arch be leſs than a Quadrant, or greater than three Quadrants; but this if contrary-wiſe;) *divided by* Rc, *be Multiplied into* $\sqrt{:} 4Rq - Aq$: *Product is equal to* D.

XCII. And therefore, $Rc \,.\, 2RqA \backsim Ac :: \sqrt{:} 4Rq - Aq \,.\, D$. That is,

XCIII. *As the Cube of the Radius, to the Solid of the Subtenſe of the ſingle Arch into the Difference of the Square of it ſelf, and of the double Square of the Radius: So is the Subtenſe of the Difference of that ſingle Arch from a Semicircumference, to the Subtenſe of the Quadruple Arch.*

XCIV. Now what was before ſaid: (at § 15, *Chap.* 29.) That the Subtenſe of an Arch, with that of its Remainder to a Semicircumference (or of its Exceſs above a Semicircumference) will require the ſame Subtenſe of the double Arch; is the ſame as to ſay, that, *From any Point of Circumference, two Subtenſes drawn to the two ends of any inſcribed Diameter,* (as A, E,) *will require the ſame Subtenſe* (B) *of the double Arch.* *Fig.* I.

XCV. And what is ſaid: (at § 12, 26, *Chap. preced.*) That the Subtenſe of an Arch leſs than a Trient, and of its Reſidue to a Trient (as A, E,) and of a Trient increaſed by either of thoſe, (as Z,) will have the ſame Subtenſe of the Triple Arch; is the ſame in effect with this, that, *From any Point of the Circumference, three ſubtenſes drawn to the three Angles of any inſcribed (Regular) Trigone* (as A, E, Z,) *will have the ſame Subtenſe* (C) *of the Triple Arch.* *Fig.* XI.

Fig. XXIII. XCVI. And what is ſaid here: (at § 18, 20.) That the Subtenſe of an Arch leſs than a Quadrant, and of its Reſidue to a Quadrant, (as A, E,) and of a Quadrant increaſed by either of theſe, (as P, S,) will have the ſame Subtenſe of the Quadruple Arch: Is the ſame with this, that, *From any Point of the Circumference, Four Subtenſes drawn to the four Angles of any inſcribed (Regular) Tetragone*, (as A, E, P, S,) *will have the ſame Subtenſe* (D) *of the Quadruple Arch.*

XCVII. But the ſame holds, reſpectively, in other Multiplications of Arches; as five Subtenſes from the ſame Point, to the five Angles of an inſcribed (Regular) Pentagon; and ſix, to the ſix Angles of an Hexagon; *&c.* Will have the ſame Subtenſe of the Arches Quintuple, Sextuple, *&c.* For they all depend on the ſame common Principle, That a Semicircumference Doubled, a Trient Tripled, a Quadrant Quadrupled, a Quintant Quintupled, a Sextant Sextupled, *&c.* Make one entire Revolution; which as to this buſineſs, is the ſame as nothing. And therefore, univerſally,

XCVIII. *From any Point of the Circumference, two, three, four, five, ſix, or more ſubtenſes, drawn to ſo many (ends of the Diameter, or) Angles of a (Regular) Polygone of ſo many Angles, however inſcribed, will have the ſame Subtenſe of the Arch Multiplied by the number of ſuch ends or Angles.* And therefore,

CXIX. *An Equation belonging to ſuch Multiplication or Section of an Arch or Angle, muſt have ſo many Roots* (Affirmative or Negative) *as is the Exponent of ſuch Multiplication or Section.* As two for the Biſection, three for the Triſection, four for the Quadriſection, five for the Quinquiſection: And ſo forth.

C. And conſequently, *Such Equations may accordingly be reſolved, by ſuch Section of an Angle.* As was before noted (at § 61, *Chap. preced.*) of the Triſection of an Angle.

CHAP.

CHAP. IV.

Of the Quintuplation and Quinquiſection of an ARCH *or* ANGLE.

I. IF in a Circle be inſcribed a Quadrilater, whoſe ſides A, F, (the Subtenſes of the ſingle Arch and the Quintuple,) be Parallel; B, B, (ſubtenſes of the double) oppoſite: The Diagonals will be C, C, (the ſubtenſes of the Triple,) as is evident from the Figure. But it is evident alſo, that, in this caſe, the ſingle Arch muſt be leſs than a Quintant (or fifth part) of the whole Circumference. *Fig.* XXIV.

II. And therefore (the Rect-angle of the Diagonals being equal to the two Rect-angles of the oppoſite ſides,) $Cq - Bq = AF$. (And by the ſame reaſons $cq - bq = EF$.) That is,

III. *The Square of the Subtenſe of the Triple Arch, wanting the Square of the Subtenſe of the double Arch, is equal to the Rect-angle of the Subtenſes of the ſingle and of the Quintuple; the ſingle Arch being leſs than a fifth part of the whole Circumference.*

IV. And therefore, if it be divided by one of them; it gives the other. That is, $\frac{Cq - Bq}{A} = F$; and $\frac{Cq - Bq}{F} = A$. (And, in like manner $\frac{cq - bq}{E} = F$; and $\frac{cq - bq}{F} = E$.

V. But $C + B$ into $C - B$ is equal to $Cq - Bq$. And therefore, $A . C + B :: C - B . F$. That is,

VI. *As the Subtenſe of the ſingle Arch (leſs than a fifth part of the whole Circumference) to the Aggregate of the ſubtenſes of the Triple and double; ſo is the Exceſs of the Subtenſe of the Triple above that of the double, to that of the Quintuple.*

VII. And becauſe (by § 8. *Chap.* 30.) $C = 3A - \frac{Ac}{Rq}$; and therefore $Cq = 9Aq - \frac{6Aqq}{Rq} + \frac{Acc}{Rqq}$. And (by § 7. *Chap.* 29.) $Bq = 4Aq - \frac{Aqq}{Rq}$: Therefore, $Cq - Bq = 5Aq - \frac{5Aqq}{Rq} + \frac{Acc}{Rqq} = AF$: And $5A - \frac{5Ac}{Rq} + \frac{Aqc}{Rqq} = F = 5E - \frac{5Ec}{Rq} - \frac{Eqc}{Rqq}$. That is,

VIII. *If, to the Quintuple of the Subtenſe of an Arch leſs than a Quintant, wanting the Quintuple of the Cube of the ſame Subtenſe divided by the Square of the Radius, be added the Quadricube* (or fifth Power) *of the ſame Subtenſe divided by the Biquadrate of the Radius; the Reſult is the Subtenſe of the Quintuple Arch.*

IX. The ſame may be otherwiſe thus evinced; taking a Quadrilater whoſe oppoſite ſides are A, A, and F, C; and the Diagonals D, D. And therefore, $Dq - Aq = CF$. (And, in like manner, $bq - Eq = cF$.) That is, *Fig.* XXV.

X. *The*

Fig. XXV. X. *The Square of the Subtenſe of the Quadruple Arch, wanting the Square of the Subtenſe of the ſingle Arch (leſs than a Quintant,) is equal to the Rect-angle of the ſubtenſes of the Triple and Quintuple.* And being divided by either of theſe, it gives the other of them.

XI. And (becauſe D+A into D—A is equal to Dq—Aq,) C . D + A :: D—A . F. (And c . d+E :: d—E . F.) That is,

XII. *As the Subtenſe of the Triple Arch, to the ſum of the ſubtenſes of the Quadruple and of the ſingle (this being leſs than a Quintant,) ſo is the Difference of theſe, to the Subtenſe of the Quintuple.*

XIII. But, (by § 7, *Chap. preced.*) $\frac{2RqA-Ac}{Rc}\sqrt{:4Rq-Aq:}=D$. And therefore, $\frac{15RccAq-20RqqAqq+8RqAcc-Accq}{Rcc}=Dq$: Which abated by Aq, leaves $15Aq-\frac{20Aqq}{Rq}+\frac{8Acc}{Rqq}-\frac{Accq}{Rcc}=Dq-Aq=CF$. And this divided by $C=\frac{3RqA-Ac}{Rq}$; gives $\frac{5RqqA-5RqAc+Acq}{Rqq}=F=\frac{5RqqE-5RqEc+Ecq}{Rqq}$. As before,

Fig. XXVI. XIV. The ſame, is a third way, thus evinced; Inſcribing a Quadrilater, whoſe oppoſite ſides are A, C, and A, F; and the Diagonals B, D. And therefore AC+AF=BD; and BD—AC=AF. (And in like manner, bd—cE=EF.) That is,

15. *The Rect-angle of the ſubtenſes of the double and Quadruple Arch, wanting that of the ſubtenſes of the ſingle (being leſs than a Quintant) and of the Triple; is equal to the Rect-angle of the ſubtenſes of the ſingle and Quintuple.* And, being divided by either of theſe, gives the other of them.

XVI. And therefore, A . B :: D . C+F. (And E . b :: d . c+F.) That is,

XVII. *As the Subtenſe of a ſingle Arch (leſs than a Quintant,) to that of the double; ſo is that of the Quadruple, to the Aggregate of the ſubtenſes of the Triple and Quintuple.*

XVIII. But $B=\frac{A}{R}\sqrt{:4Rq-Aq}$. And $D=\frac{2RqA-Ac}{Rc}\sqrt{:4Rq-Aq}$. Therefore, $BD=\frac{2RqAq-Aqq}{Rqq}$ into $4Rq-Aq:=\frac{8RqqAq-6RqAqq+Acc}{Rqq}$. Likewiſe, $C=3A-\frac{Ac}{Rq}$; and therefore, $AC=\frac{3RqAq-Aqq}{Rq}=\frac{3RqqAq-RqAqq}{Rqq}$. And therefore, $BD-AC=\frac{5RqqAq-5RqAqq+Acc}{Rqq}=AF$. And $\frac{5RqqA-5RqAc+Acq}{Rqq}=F=\frac{5RqqE-5RqEc+Ecq}{Rqq}$. As before,

XIX. Or, we may thus compute it: Becauſe $AC+AF=BD=\frac{8RqqAq-6RqAqq+Acc}{Rqq}$ (as before;) therefore $\frac{BD}{A}=\frac{8RqqA-6RqAc+Acq}{Rqq}=C+F$. And therefore, (ſubtracting $C=3A-\frac{Ac}{Rq}$) $\frac{5RqqA-5RqAc+Acq}{Rqq}=F=\frac{5RqqE-5RqEc+Acq}{Rqq}$. As before,

XX. *The*

XX. The ſame way, a fourth way, be thus evinced; Inſcribing a Quadrilater whoſe oppoſite ſides are B, F, and B, A; and the Diagonals C, D. And therefore $BA+BF=CD$, and $CD-BA=BF$. (And, in like manner, $cb-bE=BF$.) And $\frac{CD-BA}{B}=F$. That is, *Fig.* XXVII.

XXI. *The Rect-angle of the ſubtenſes of the Treble and Quadruple Arches, wanting that of the ſubtenſes of the double and ſingle (this being leſs than a Quintant,) is equal to that of the ſubtenſes of the double and Quintuple.* And being divided by the one, it gives the other.

XXII. And therefore, $B . C :: D . A+F$. (And $b . c :: b . E+F$.) That is,

XXIII. *As the ſubtenſe of the double Arch, to that of the Triple, ſo is that of the Quadruple, to the Aggregate of the ſubtenſes of the ſingle (being leſs than a Quintant) and of the Quintuple.*

XXIV. But $C=\frac{3RqA-Ac}{Rq}$; and $D=\frac{2RqA-Ac}{Rc}\sqrt{:}\,4Rq-Aq$. Therefore, $CD=\frac{6RqqAq-5RqAqq+Acc}{Rcq}\sqrt{:}\,4Rq-Aq$. Likewiſe, $B=\frac{A}{R}\sqrt{:}\,4Rq-Aq$: And therefore, $BA=\frac{Aq}{R}\sqrt{:}\,4Rq-Aq$. And conſequently, $CD-BA=BF=\frac{5RqqAq-5RqAqq+Acc}{Rcq}\sqrt{:}\,4Rq-Aq$. And (dividing by $B=\frac{A}{R}\sqrt{:}\,4Rq-Aq$:) $\frac{5RqqA-5RqAc+Acq}{Rqq}=F=\frac{5RqqE-5RqEc+Ecq}{Rqq}$. As before,

XXV. Or, we may thus compute it: Becauſe $BA+BF=CD=\frac{6RqqAq-5RqAqq+Acc}{Rcq}\sqrt{:}\,4Rq-Aq$: Therefore, (dividing by $B=\frac{A}{R}\sqrt{:}\,4Rq-Aq$;) $\frac{CD}{B}=\frac{6RqqA-5RqAc+Acq}{Rqq}=A+F$. And $\frac{5RqqA-5RqAc+Acq}{Rqq}=F$. As before,

XXVI. Or thus; becauſe $BA+BF=CD$; and therefore, $\frac{D}{B}C=A+F$: And alſo, $D=\frac{2RqA-Ac}{Rc}\sqrt{:}\,4Rq-Aq$: And $B=\frac{A}{R}\sqrt{:}\,4Rq-Aq$: Therefore, $\frac{D}{B}=\frac{2Rq-Aq}{Rq}$; and this Multiplied by $C=\frac{3RqA-Ac}{Rq}$, makes $\frac{DC}{B}=\frac{6RqqA-5RqAc+Acq}{Rqq}=A+F$: And $\frac{5RqqA-5RqAc+Acq}{Rqq}=F$. As before. *Fig.* XXVII.

XXVII. But if the Arch to be Quintupled be juſt the fifth part of the whole Circumference, (and conſequently the Quintuple Arch one intire Revolution;) the Subtenſe of that Quintuple will vaniſh, or become equal to nothing.

XXVIII. And

Fig. XXVII. XXVIII. And therefore, in this caſe, $\frac{5RqqA-5RqAc+Acq}{Rqq} = F = 0$. And ſo $5RqqA-5RqAc+Acq=0$; and $5Rqq-5RqAq+Aqq=0$; or, $5Rqq=5RqAq-Aqq$; or, $5Rq=(5Aq-\frac{Aqq}{Rq}=)\frac{5RAq}{R}=\frac{Aqq}{Rq}$. Which is a Quadratick Equation, whoſe Root is $\frac{Aq}{R}$, and the Co-efficient of the middle Term $5R$, and the abſolute quantity $5Rq$.

XXIX. Therefore, (by reſolving the Equation) $\frac{1}{2}R\pm\sqrt{:\frac{1}{4}Rq-5Rq:}=\frac{1}{2}R\pm\sqrt{\frac{1}{4}Rq}=\frac{5\pm\sqrt{5}}{2}R=\frac{Aq}{R}$.

XXX. Of which ambiguous Equation, the leſſer Root is to be choſen, That is, $\frac{5-\sqrt{5}}{2}Rq=Aq$; and therefore, $R\sqrt{\frac{5-\sqrt{5}}{2}}(=R\sqrt{\frac{1}{2}\times\sqrt{5}-1})=A$, the Subtenſe of a Quintant. That is,

XXXI. *The Radius Multiplied into* $\sqrt{\frac{5-\sqrt{5}}{2}}$, *is equal to the Subtenſe of a Quintant, or of* 72 *Degrees.*

Fig. XXVIII. XXXII. The ſame may be otherwiſe thus inferred: If, in a Circle, be inſcribed a Regular Pentagon; whoſe ſide A ſhall be reputed as the Subtenſe of a ſingle Arch: It's evident that the Subtenſe of the Duple, and of the Triple, will be the ſame. (For the ſame Chord which on the one ſide, ſubtends the Duple, doth on the other ſide, ſubtend the Triple.) And therefore, $\frac{A}{R}\sqrt{:4Rq-Aq:}=B=C=\frac{3RqA-Ac}{Rq}$. And $\sqrt{:4Rq-Aq:}=\frac{3Rq-Aq}{R}$. And $4Rq-Aq=(\frac{9Rqq-6RqAq+Aqq}{Rq}=)\ 9Rq-6Aq+\frac{Aqq}{Rq}$. And therefore, $5Rq-5Aq+\frac{Aqq}{Rq}=0$. And therefore, (as before) $5Rq=5Aq-\frac{Aqq}{Rq}$; and ſo onward as above.

XXXIII. Now becauſe (as is already ſhew'd) $\sqrt{:4Rq-Aq:}=\frac{3Rq-Aq}{R}=3R-\frac{Aq}{R}$: This therefore will be the Subtenſe of a Seſquiquintant (or one Quintant and an half, or three tenth parts,) that is, of 108 Degrees: As being that Arch which with the Quintant doth complete the Semicircumference. That is,

XXXIV. *The Difference of the Squares of the Subtenſes of the Trient, and of the Quintant, divided by the Radius; is equal to the Subtenſe of the Seſquiquintant, or* 108 *Degrees.* (For $3Rq$ is the Square of the Subtenſe of the Trient; and Aq, of the Quintant; and the Difference of theſe $3Rq-Aq$ divided by the Radius, is the Subtenſe.) Or thus,

XXXV. *If from the Triple of the Radius* $3R$, *be ſubducted the Square of the Subtenſe of a Quintant divided by the Radius; the Remainder is the Subtenſe of a Seſquiquintant, or* 108 *Degrees.* $3R-\frac{Aq}{R}$.

Fig. XXVIII. XXXVI. But the Square of the Subtenſe of a Quintant ſo divided, is (as before) $\frac{Aq}{R}=\frac{5-\sqrt{5}}{2}R$; which therefore ſubtracted from $3R$, leaves $\frac{1+\sqrt{5}}{2}R$ the Subtenſe of 108 Degrees.

XXXVII. Now,

XXXVII. Now, if the Radius be cut in extream and mean proportion, the greater Segment thereof is $\frac{\sqrt{5}-1}{2}$ R (by 11. *El.* 2.) to which if 1 R be added, we have $\frac{\sqrt{5}+1}{2}$ R (the Subtenſe of 108 Degrees as before;) And therefore, *Fig.* XXVIII.

XXXVIII. *If the Radius being cut in extreme and mean proportion, the greater Segment thereof, be added to the whole Radius; the ſum is equal to the Subtenſe of* 108 *Degrees.*

XXXIX. Yet again; If, of a Pentagone ſo inſcribed, the ſide A be conſidered as the Subtenſe of a ſingle Arch; the ſame will alſo be the Subtenſe of the Quadruple. (For the ſame Chord ſubtends on the one ſide to one Quintant, and on the other ſide to four ſuch.)

XL. And therefore, in this caſe, $A = D = \frac{2RqA - Ac}{Rc} \sqrt{:} 4Rq - Aq$. And $Rc A = 2RqA - Ac$ into $\sqrt{:} 4Rq - Aq$. That is, $Rc = 2Rq - Aq$, into $\sqrt{:} 4Rq - Aq$. And (the Square hereof) $Rcc = 16Rcc - 20RqqAq + 8RqAqq - Acc$; or $15Rcc - 20RqqAq + 8RqAqq - Acc = 0$.

XLI. Now this laſt Equation, if divided by $3Rq - Aq = 0$, will afford this Equation; $5Rqq - 5RqAq + Aqq = 0$.

$$
\begin{array}{l|l}
3Rq-Aq=0) & 15Rcc-20RqqAq+8RqAqq-Acc=0\ (5Rqq-5RqAq+Aqq=0 \\
 & 15Rcc-5RqqAq \\
\hline
 & -15RqqAq+8RqAqq-Acc \\
 & -15RqqAq+5RqAqq \\
\hline
 & +3RqAqq-Acc \\
 & +3RqAqq-Acc \\
\hline
 & 00 \quad 00
\end{array}
$$

XLII. And therefore $3Rq = Aq$; is one of the Plain Roots of that Equation. And therefore, $R\sqrt{3} = A$, which is the Subtenſe of a Trient. (Which is true, becauſe alſo the Quadruple of a Trient, hath the ſame Subtenſe with the ſingle Trient.)

XLIII. But there are alſo two other Plain Roots included in the Reſulting Equation $5Rqq - 5RqAq + Aqq = 0$; or $5Rqq = 5RqAq - Aqq$. For,

XLIV. The leſſer of them is $\frac{1}{2}Rq - \sqrt{:} \frac{1}{4}Rqq - \frac{2}{4}Rqq := \frac{5-\sqrt{5}}{2}$ $Rq = Aq$; the Square of the Subtenſe of a Quintant. As before,

XLV. The greater of them is $\frac{1}{2}Rq + \sqrt{:} \frac{1}{4}Rqq - \frac{2}{4}Rqq := \frac{5+\sqrt{5}}{2}$ $Rq = Aq$; the Square of the Subtenſe of a double Quintant, or of a Triple, (as we ſhall ſee afterward) that is, of 144, or of 216 Degrees. For the Quadruple of theſe alſo, will have the ſame Subtenſe with that of the ſingle. For $\frac{2}{5} \times 4 = \frac{8}{5} = 1 + \frac{3}{5}$. And $\frac{3}{5} \times 4 = \frac{12}{5} = 2 + \frac{2}{5}$. Where the Exceſs above the entire Revolutions (which are here Equivalent to nothing) is $\frac{2}{5}$, or $\frac{3}{5}$, both which have the ſame Subtenſe (as at § 32.) over that of the ſingle Arch; that is, $4\frac{2}{5}$, or $\frac{3}{5}$.

Fig. XXVIII. XLVI. Since therefore (as is ſhewed) $\frac{5-\sqrt{5}}{2}Rq=Aq$, is the Square of the Subtenſe of a Quintant; the Square of the Subtenſe of its Reſidue to the Semicircumference muſt be $4Rq-\frac{5-\sqrt{5}}{2}Rq=\frac{3+\sqrt{5}}{2}Rq$. Which is therefore the Square of the Subtenſe of 108 (=180—72.) And the Quadratick Root thereof $\frac{\sqrt{5}+1}{2}R$ (=$\sqrt{:\frac{3+\sqrt{5}}{2}Rq:}$) as was alſo before ſhewed.

XLVII. And for as much as $\frac{\sqrt{5}+1}{2}R$ is the Subtenſe of 108 Degres, that is of 18 Degrees above a Quadrant; let this Subtenſe be S, and the Subtenſe of 18 Degrees, (which is the Exceſs above a Quadrant) E. Therefore, (by § 54, *Chap. preced.*) $Sq+Eq-SE\sqrt{2}=2Rq$. And therefore, $Sq-2Rq=ES\sqrt{2}-Eq$. And (by reſolving that Equation) $\frac{1}{2}S\sqrt{2}\pm\sqrt{:2Rq-\frac{1}{2}Sq:}=E$. The leſſer of which Roots is here to be choſen, becauſe E is the leſſer of the two S, E.

XLVIII. But (as is ſhewed) $S=\frac{\sqrt{5}+1}{2}R$, and therefore, $\frac{1}{2}S\sqrt{2}=\frac{\sqrt{10}+\sqrt{2}}{4}R$: And $Sq=\frac{3+\sqrt{5}}{2}Rq$, and therefore, $2Rq-\frac{1}{2}Sq=\frac{5-\sqrt{5}}{4}Rq$, (half the Square of the Subtenſe of a Quintant,) whoſe Square Root is $\frac{\sqrt{:5-\sqrt{5}:}}{2}R$. And therefore, (the leſs Root being here of uſe) $\frac{\sqrt{10}+\sqrt{2}-2\sqrt{:5-\sqrt{5}:}}{4}R=E$, the Subtenſe of 18 Degrees.

XLIX. The ſame Arch of 18 Degrees, is alſo the Complement of a Quintant to a Quadrant. And therefore if the Subtenſe of a Quintant (or 72 Degrees, being leſs than a Quadrant,) be called $A=R\sqrt{\frac{5-\sqrt{5}}{2}}$; and the Subtenſe of its Complement to a Quadrant (or of 18 Degrees) E: Then (by § 52. *Chap. preced.*) $Aq+Eq+AE\sqrt{2}=2Rq$. And therefore, $Eq+AE\sqrt{2}=2Rq-Aq$. And (reſolving the Equation,) $\sqrt{:\frac{1}{4}Aq+2Rq-Aq:}(=\sqrt{:2Rq-\frac{1}{2}Aq:})-\frac{1}{2}A\sqrt{2}=E$.

L. But $Aq=\frac{5-\sqrt{5}}{2}Rq$, and therefore, $2Rq-\frac{1}{2}Aq=\frac{3+\sqrt{5}}{4}Rq$ (half the Square of the Subtenſe of the Seſquiquintant, or 108 Degrees;) and the Square Root thereof $\sqrt{\frac{3+\sqrt{5}}{4}Rq}=\frac{\sqrt{10}+\sqrt{2}}{4}R$. And $\frac{1}{2}A\sqrt{2}=A\sqrt{\frac{1}{2}}=R\sqrt{\frac{5-\sqrt{5}}{4}}=\frac{\sqrt{:5-\sqrt{5}:}}{2}R$. And therefore, $\frac{\sqrt{10}+\sqrt{2}-2\sqrt{:5-\sqrt{5}:}}{4}R=E$, the Subtenſe of 18 Degrees, as before. That is,

LI. *The Subtenſe of the Seſquiquintant, or of* 108 *Degrees, (that is, the greater Segment of the Radius cut in extream and mean proportion, increaſed by the entire Radius,) Multiplied into* $\sqrt{2}$ (for $\frac{\sqrt{5}+1}{2}R\sqrt{2}=\frac{\sqrt{10}+\sqrt{2}}{2}R$,) *wanting the Subtenſe of the Quintant Multiplied alſo into* $\sqrt{2}$ (for $R\sqrt{\frac{5-\sqrt{5}}{2}}$ into $\sqrt{2}$, $=R\sqrt{:5-\sqrt{5}:}$) *is equal to the double of the Subtenſe of* 18 *Degrees.* (And half thereof, equal to that Subtenſe.) Or,

LII. *The Difference of the Subtenſes of the Seſquiquintant and of the Quintant,* (or of 108 Degrees, and of 72 Degrees) *divided by* $\sqrt{2}$, *is equal to the Subtenſe of* 18 *Degrees.* That is, that Difference is double in Power to this Subtenſe, *(duplum poteſt,)* or, the Square of that, is double to the Square of this.

LIII. But,

LIII. But, *The ſubtenſes of the Quintant and Seſquiquintant*, (that is, of 72, and of 108 Degrees, which together complete the Semicircumference) *Multiplied the one into the other*, (or the Rectangle of them,) *divided by the Radius*; *is equal to the Subtenſe of the double Arch of either.* For, by § 9, *Chap.* R) AE (B. That is, of 144, or of 216 Degrees. That is, of the double, or Triple Quintant, (theſe two having the ſame Subtenſe.) That is, $\frac{1+\sqrt{5}}{2}\sqrt{:}\frac{5-\sqrt{5}}{2}\,Rq := R\sqrt{\frac{3+\sqrt{5}}{2}\times\frac{5-\sqrt{5}}{2}} = R\sqrt{\frac{5+\sqrt{5}}{2}}$. That is, *Fig.* XXVIII.

LIV. *The Radius Multiplied into* $\sqrt{\frac{5+\sqrt{5}}{2}}$, *is equal to the Subtenſe of the Biquintant, and of the Triquintant*; That is, *to the Subtenſe of* 144, *and of* 216 *Degrees.*

LV. And the Square of this ſubtracted from the Square of the Diameter, leaves $\left(\frac{3-\sqrt{5}}{2}\,Rq\right)$ the Square of the Subtenſe of 36 Degrees; (as being what 144 Degrees wants of a Semicircumference, and what 216 exceeds it. For $180-144=36=216-180$.) And the Square Root thereof is that Subtenſe, $\sqrt{:}\frac{3-\sqrt{5}}{2}\,Rq := \frac{\sqrt{5}-1}{2}R$. That is,

LVI. *The greater Segment of the Radius cut in extream and mean Proportion, is the Subtenſe of* 36 *Degrees.* That is, of half a Quintant, or the ſide of the inſcribed Decagon.

LVII. But we had before ſhewn (at § 38.) that this added to the Radius (which is the Subtenſe of 60 Degrees, or ſide of the inſcribed Hexagon,) is equal to the Subtenſe of 108 Degrees, or Seſquiquintant: Therefore,

LVIII. *The Aggregate of the ſubtenſes of* 36 *Degrees, and of* 60 *Degrees*, (that is, the ſides of the inſcribed Decagon and Hexagon,) *is equal to that of* 108 *Degrees*; (that is, of the Seſquiquintant, or three Tenths.)

LIX. If therefore to the Subtenſe of 36 Degrees, $\frac{\sqrt{5}-1}{2}R$, be added that of 108 Degrees $\frac{\sqrt{5}+1}{2}R$, it makes $\sqrt{5}\,Rq$, or $R\sqrt{5}$. That is,

LX. *The Subtenſe of the Semiquintant* (or 36 Degrees) *and of the Seſquiquintant* (or 108 Degrees) *added together, are in power Quintuple to the Radius*, (that is, the Square of that Aggregate is equal to five Squares of the Radius.) For, $\frac{\sqrt{5}-1}{2}R+\frac{\sqrt{5}+1}{2}R = R\sqrt{5}$.

LXI. *And their Difference is equal to the Radius.* For, $\frac{\sqrt{5}+1}{2}R-\frac{\sqrt{5}-1}{2}R = R$.

LXII. *And the Rectangle of them, is equal to the Square of the Radius.* For, $\frac{\sqrt{5}+1}{2}R\times\frac{\sqrt{5}-1}{2}R = Rq$.

LXIII. *And the ſum of their Squares is Triple to the Square of the Radius.* (Or, *equal to the Square of the ſide of the inſcribed Trigone.*) That is, $\frac{3+\sqrt{5}}{2}Rq+\frac{3-\sqrt{5}}{2}Rq = 3Rq$.

Fig. XXVIII.

LXIV. *And the Difference of their Squares, is in Power Quintuple to the Square of the Radius,* (or, *equal to five ſquared Squares of the Radius.* For, $\frac{3+\sqrt{5}}{2}$ Rq—) $\frac{3-\sqrt{5}}{2}$ Rq = Rq $\sqrt{5}$ = $\sqrt{5}$ Rqq.

LXV. Again, *The ſum of the Squares of the ſubtenſes of the Quintant and Biquintant* (or of 72 Degrees, and of 144 Degrees,) *is Quintuple to the Square of the Radius.* For, $\frac{5-\sqrt{5}}{2}$ Rq + $\frac{5+\sqrt{5}}{2}$ Rq = 5 Rq.

LXVI. *And the Difference thereof, is in Power Quintuple to the Biquadrate of the Radius.* For, $\frac{5+\sqrt{5}}{2}$ Rq — $\frac{5-\sqrt{5}}{2}$ Rq = Rq $\sqrt{5}$ = $\sqrt{5}$ Rqq.

LXVII. *And the Rectangle of them, is Quintuple of the Biquadrate of the Radius.* For, $\frac{5-\sqrt{5}}{2}$ Rq × $\frac{5+\sqrt{5}}{2}$ Rq = 5 Rqq.

LXVIII. We have therefore (as hath been ſeverally demonſtrated) theſe ſubtenſes, anſwering to their ſeveral Arches, or portions of the whole Circumferences, *viz.*

Subtenſes.	Degrees.	Parts of the whole.
$\frac{\sqrt{5}-1}{2}$ R	36 . 324	$\frac{1}{10}$. $\frac{2}{10}$
R $\sqrt{\frac{5-\sqrt{5}}{2}}$	72 . 288	$\frac{3}{10}$. $\frac{4}{10}$
$\frac{\sqrt{5}+1}{2}$ R	108 . 252	$\frac{3}{10}$. $\frac{7}{10}$
R $\sqrt{\frac{5+\sqrt{5}}{2}}$	144 . 216	$\frac{4}{10}$. $\frac{9}{10}$
2 R	180	$\frac{1}{10}$

LXIX. By the like method we may find the ſubtenſes of $\frac{1}{20}$, $\frac{3}{20}$, $\frac{7}{20}$, $\frac{9}{20}$, of the whole Circumference: (as likewiſe of $\frac{12}{20}$, $\frac{17}{20}$, $\frac{13}{20}$, $\frac{11}{20}$,) For the Reſidue of $\frac{1}{5}$ to a Quadrant (or Exceſs of $\frac{1}{10}$ above a Quadrant) is $\frac{1}{20}$; and therefore the Subtenſe thereof is $\frac{\sqrt{10+\sqrt{2}}-2\sqrt{:5-\sqrt{5}:}}{4}$ R, (as is ſhewed before at § 48.) or (which is Equivalent) $\frac{\sqrt{:3+\sqrt{5}:}-\sqrt{:5-\sqrt{5}}}{2}$ R. Or, R $\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}$. And the Reſidue of this to the Semicircumference is $\frac{9}{20}$; whoſe Subtenſe therefore is R $\sqrt{\frac{4+\sqrt{:10+2\sqrt{5}:}}{2}}$. Or, R $\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}$. Again, the Reſidue of $\frac{1}{10}$ to a Quadrant (or the Exceſs of $\frac{3}{5}$ above a Quadrant,) is $\frac{3}{20}$; whoſe Subtenſe therefore is $\frac{\sqrt{:5+\sqrt{5}:}-\sqrt{:3-\sqrt{5}:}}{2}$ R. Or, $\frac{2\sqrt{:5+\sqrt{5}:}-\sqrt{10+\sqrt{2}}}{4}$ R. And the Reſidue of this to a Semicircumfer ence is $\frac{7}{20}$; whoſe Subtenſe therefore is R $\sqrt{\frac{4+\sqrt{:10-\sqrt{5}:}}{2}}$. Or, R $\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}$. For, in ſuch caſes, the ſame value may be expreſſed in very different ways. (All which may be eaſily proved by computation, in like manner as thoſe before going; and like Corollaries eaſily deduced from them.) And the Remainders of theſe to the whole Circumferences ($\frac{12}{20}$, $\frac{17}{20}$, $\frac{11}{20}$, $\frac{11}{20}$,) have the ſame Subtenſes with them.

LXX. We have therefore, now, theſe Subtenſes, for the Arches and portions following.

Degrees

Degrees of Arches.	Portions of the whole.	Subtenſes.	*Fig.* XXVIII.
0 . 360	$\frac{0}{20}$. $\frac{20}{20}$	0	
18 . 342	$\frac{1}{20}$. $\frac{19}{20}$	$\frac{\sqrt{:3+\sqrt{5}:}-\sqrt{:5-\sqrt{5}:}}{2}$ R	
36 . 324	$\frac{2}{20}$. $\frac{18}{20}$	$\frac{\sqrt{5}-1}{2}$ R	
54 . 306	$\frac{3}{20}$. $\frac{17}{20}$	$\frac{\sqrt{:5+\sqrt{5}:}-\sqrt{:3-\sqrt{5}:}}{2}$ R	
72 . 288	$\frac{4}{20}$. $\frac{16}{20}$	$R\sqrt{\frac{5-\sqrt{5}}{2}}$	
90 . 270	$\frac{5}{20}$. $\frac{15}{20}$	$R\sqrt{2}$	
108 . 252	$\frac{6}{20}$. $\frac{14}{20}$	$\frac{\sqrt{5}+1}{2}$ R	
126 . 234	$\frac{7}{20}$. $\frac{13}{20}$	$R\sqrt{\frac{4+\sqrt{:10-2\sqrt{5}:}}{2}}$	
144 . 216	$\frac{8}{20}$. $\frac{12}{20}$	$R\sqrt{\frac{5+\sqrt{5}}{2}}$	
162 . 198	$\frac{9}{20}$. $\frac{11}{20}$	$R\sqrt{\frac{4+\sqrt{:10+2\sqrt{5}:}}{2}}$	
180	$\frac{10}{20}$	2 R	

LXXI. Now if all theſe Arches be compared with the Trient; and the Sums and Differences of them ſo compared be obſerved: We ſhall thence have a great many more Subtenſes, by what is before delivered, at § 15, 47, *Chap.* 30. As for Example,

LXXII. Suppoſe the Subtenſe of 72 Degrees to be $A = R\sqrt{\frac{5-\sqrt{5}}{2}}$, and the Subtenſe of 48 (= 120 − 72) to be E. Then, (by § 15, *Chap.* 30.) $Aq + AE + Eq = 3Rq$; and therefore, $AE + Eq = 3Rq - Aq$. And (by reſolving the Equation) $\sqrt{:\frac{1}{4}Aq + 3Rq - Aq:} (= \sqrt{:3Rq - \frac{1}{4}Aq:}) - \frac{1}{2}A = E$.

LXXIII. But $A = R\sqrt{\frac{5-\sqrt{5}}{2}}$, and $Aq = \frac{5-\sqrt{5}}{2}Rq$. Therefore, $E = \sqrt{:3Rq - \frac{1}{4}Aq:} - \frac{1}{2}A = \sqrt{:\frac{9+3\sqrt{5}}{8}Rq:} - \sqrt{:\frac{5-\sqrt{5}}{8}Rq:} = \frac{\sqrt{:9+3\sqrt{5}:}-\sqrt{:5-\sqrt{5}:}}{2\sqrt{2}}$ $R = \frac{\sqrt{:18+6\sqrt{5}:}-\sqrt{:10-2\sqrt{5}:}}{4}R = \frac{\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}R$: the Subtenſe of 48 Degrees, and therefore alſo of 312 Degrees.

LXXIV. In like manner: Suppoſe (as before) A the Subtenſe of 72 Degrees; and Z the Subtenſe of 192 (= 120 + 72.) Then (by § 47, *Chap.* 30.) $Zq - AZ + Aq = 3Rq$; and $Zq - AZ = 3Rq - Aq$. And (reſolving the Equation) $\sqrt{:3Rq - \frac{3}{4}Aq:} + \frac{1}{2}A = Z = \sqrt{:\frac{9+3\sqrt{5}}{8}Rq:} + \sqrt{:\frac{5-\sqrt{5}}{8}Rq:} = \frac{\sqrt{:18+6\sqrt{5}:}+\sqrt{:10-2\sqrt{5}:}}{4}R = \frac{\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}R$. The Subtenſe of 192 Degrees, and therefore alſo of 168 Degrees. That is,

LXXV. *If the Subtenſe of a Trient, be increaſed by the greater Segment of ſuch Subtenſe cut in extream and mean proportion; And thereunto be Added, or taken from it, the Subtenſe of a Quintant: The reſult is, in caſe of Addition, the double Subtenſe of* 168 *and of* 192 *Degrees; in caſe of Subtraction, the double Subtenſe of* 48 *and of* 312 *Degrees.* Or thus,

Fig. XXVIII. LXXVI. *If to the greater Segment of the Subtenſe of a Trient (cut in extream and mean Proportion,) be added the Sum, or Difference of the ſubtenſes of the Trient and of the Quintant: The Reſult is, the double Subtenſe, in the firſt caſe, of* 168 *and of* 192 *Degrees; in the latter caſe, of* 48 *and of* 312 *Degrees.* For, $\surd : \frac{9+3\surd 5}{8} \text{R}q := \surd : \frac{3+\surd 5}{8} 3\text{R}q := \frac{1}{2}\surd : \frac{3+\surd 5}{2} 3\text{R}q := \frac{\surd 5+1}{4=2\times 2}\surd 3\text{R}q$, is the half of the Subtenſe of a Trient ($\surd 3\text{R}q$) increaſed by its greater Segment if ſo cut. And $\surd : \frac{5-\surd 5}{8}\text{R}q := \frac{1}{2}\text{R}\surd \frac{5-\surd 5}{2}$, is half the Subtenſe of a Quintant.

LXXVII. And the Squares of theſe ſubtenſes, ſubtracted from $4\text{R}q$, give us the Squares of the ſubtenſes of their Differences from a Semicircumference. That is, of 12 and of 132 Degrees (whereby 168 and 48 come ſhort of a Semicircumference; and whereby 192 and 312 exceed it.) For $12 = 180 - 168 = 192 - 180$; and $132 = 180 - 48 = 312 - 180$.

LXXVIII. Again, ſuppoſe the Subtenſe of a Biquintant, or 144 Degrees, (which is alſo the Subtenſe of a Triquintant, or 216 Degrees) being greater than a Trient, to be $Z = \text{R}\surd \frac{5+\surd 5}{2}$ And the Subtenſe of $96 = 216 - 120 = 240 - 144$, to be A: And the Subtenſe of $24 = 144 - 120 = 240 - 216$, to be E. Therefore, (by § 47, *Chap.* 30.) $Zq - ZA + Aq = Zq - ZE + Eq = 3\text{R}q$; and ($Zq$ being greater than $3\text{R}q$,) $Zq - 3\text{R}q = ZA - Aq = ZE - Eq$. And reſolving the Equation) $\frac{1}{2}Z \pm \surd : \frac{1}{4}Zq - Zq + 3\text{R}q := \frac{1}{2}Z \pm \surd : 3\text{R}q - \frac{1}{4}Zq : (= \text{A or E}) = \frac{1}{2}\text{R}\surd \frac{5+\surd 5}{2} \pm \frac{1}{2}\text{R}\surd \frac{9-3\surd 5}{2} = \frac{\surd : 5+\surd 5 : \pm \surd : 9-3\surd 5 :}{2\surd 2}\text{R} = \frac{\surd : 10+2\surd 5 : \pm \surd : 18-6\surd 5 :}{4}\text{R}$, the Subtenſe of 96 Degrees if connected by +; or of 24 Degrees, if by —. (That is, $\frac{\surd : 10+2\surd 5 : + \surd 15-\surd 3}{4}\text{R}$, in the firſt caſe; and $\frac{\surd : 10+2\surd 5 : -\surd 15+\surd 3}{4}\text{R}$, in the latter.) That is,

LXXIX. *If to the Subtenſe of a Biquintant be Added, or taken from it, the greater Segment of the Subtenſe of a Trient (cut in extream and mean Proportion;) it gives, in the firſt caſe, the Subtenſe of* 96, *and of* 264 *Degrees; in the latter, that of* 24, *and of* 336 *Degrees.*

LXXX. And by theſe again (by ſubducting the Squares of their ſubtenſes from $4\text{R}q$) we have (the Squares of) the ſubtenſes of their Difference from a Semicircumference, whether in Exceſs or defect. As of $84 = 180 - 96 = 264 - 180$, and of $156 = 180 - 24 = 336 - 180$.

LXXXI. And if in like manner we compare alſo the reſt of thoſe at § 70, with the Subtenſe of a Trient; we ſhall thence have the ſubtenſes of theſe Arches,

Degrees of Arches.		Of the Reſidue to the Semicircumference.	Of the Reſidues to the whole Circumference.	
120 — 108 =	12	168	348 .	192
120 — 90 =	30	150	330 .	210
120 — 72 =	48	132	312 .	228
120 — 54 =	66	114	294 .	246
120 — 36 =	84	96	276 .	264
120 — 18 =	102	78	258 .	282
120 ± 0 =	120	60	240 .	300
120 + 18 =	138	42	222 .	318
120 + 36 =	156	24	204 .	336
120 + 54 =	174	6	186 .	354

LXXXII. So

LXXXII. So that (by theſe here, and thoſe at § 70.) we have ſubtenſes for every ſixth Degree of the whole Circumference: And conſequently the Right lines (as being the half of thoſe ſubtenſes) for every third Degree of the Semicircumference. And this by the Solution of Quadratick Equations only, without the help of Cubicks or Superior Equations. And between theſe may in like manner, be interpoſed as many more as we pleaſe, by the continual Biſection of Arches. *Fig.* XXVIII.

LXXXIII. We return now to purſue the former Inquiſition which hath been intermitted. The Equation formerly propoſed at § 7, for the Quinquiſection of an Arch, $5A - \frac{5Ac}{Rq} + \frac{Acq}{Rqq} = F$; beſide the two primary Roots A and E, contains yet three other Roots, (by § 98, 99, *Chap. preced.*) anſwering to three other Chords drawn (from the ſame Point with A and E) to three other Angles of the inſcribed Pentagon: Which we ſhall call L, M, N: Whereof L ſubtends a Quintant increaſed by the Arch of A, (or three Quintants increaſed by the Arch of E;) N ſubtends a Quintant increaſed by the Arch of E, (or three Quintants with the Arch of A;) M ſubtends two Quintants increaſed by either of thoſe Arches A or E. For every of theſe Arches, if Quintupled, will have the ſame Subtenſe (of the Quintuple) F, as well as the Quintuple of the Arches A or E. *Fig.* XXIX.

LXXXIV. Of theſe three, (in caſe A and E be ſuppoſed Affirmative Roots,) L and N will be Negative; but M, Affirmative. But contrarywiſe, in caſe A and E be ſuppoſed Negative: For then L, N, will be Affirmative, and M Negative. For,

LXXXV. *When the ſingle Arch is leſs than a Quintant (or greater than four Quintants;) or when it is greater than Two, but leſs than Three; the Subtenſe of the Triple Arch will be greater than that of the double.* (As is eaſie to apprehend, or may be proved if need be, in like manner as we have formerly done in like caſes; as is after ſhewed at § 93, *&c.*) *Fig.* XXX.

LXXXVI. And therefore, if A (or E,) be the Subtenſe of the ſingle Arch; then $Cq - Bq = AF$, (or $cq - bq = EF$,) will be an Affirmative quantity, (as at § 2.)

LXXXVII. And, in like manner, if M be the Subtenſe of the ſingle, (greater than Two Quintants, but leſs than Three,) $Cq - Bq = MF$, will be alſo Affirmative. *Fig.* XXX.

LXXXVIII. *But when the ſingle Arch is greater than a Quintant, but leſs than Two, or greater than Three, but leſs than Four: The Subtenſe of the double will be greater than that of the Triple.*

LXXXIX. And therefore, if the Subtenſe of the ſingle be N, then $Cq - Bq = NF$, will be Negative. *Fig.* XXXI.

XC. And in like manner, if it be L; then $Cq - Bq = LF$ will be alſo Negative. *Fig.* XXXII.

XCI. That therefore, L, N, may have Affirmative values (as well as F,) we muſt put the Equations thus, $Bq - Cq = NF$; and $Bq - Cq = LF$. And, if ſo; then the value of the other three Roots will be Negative.

XCII. *But if the ſingle Arch be juſt a Quintant* (or two, three or more Quintants) *the Subtenſe of the double will be equal to that of the Triple.* And therefore, (putting V or X, for the Subtenſe of the ſingle,) $Cq \backsim Bq = VF = o$; or $Cq \backsim Bq = XF = o$. The Subtenſe of the Quintuple (in this caſe) vaniſhing to nothing. *Fig.* XXXIII.

XCIII. Now

Fig. XXIX. XCIII. Now that, for the Arches A, E, M, the Subtenſe of the Triple is greater (or at leaſt not leſs) than that of the Duple; but contrarywiſe for the Arches L, N; is eaſie to apprehend upon a little conſideration. For if the Arch A, or E, be 10 Degrees; B is 20; C, 30: If that be 20; B is 40; C, 60: If A be 40; B is 80; C, 120: If A be 60; B is 120; C, 180. (And hitherto is no doubt, becauſe we are not yet paſt a Semicircumference; and, till then, as the Arches increaſe, the Chords increaſe alſo; though not when we are paſt 180 Degrees.) If A be 70; B is $140 = 180 - 40$; C, $210 = 180 + 30$. So that yet the Chord of C, though paſt a Semicircumference, is greater than that of B, becauſe nearer to a Semicircumference, or 180 Degrees; (for this doth leſs exceed it, than that wants of it.) And ſo 'till we come to 72 Degrees, (or $\frac{2}{5}$ of the whole) for then B is $144 = 180 - 36$, and C, $216 = 180 + 36$; where the diſtance is equal, and accordingly the Chord of the Triple equal to that of the double. But when we be paſt a Quintant, that of the Triple becomes leſs; for if the ſingle Arch N $= \frac{1}{5} + E$ be 73; B is $146 = 180 - 34$; C, $219 = 180 + 39$; and this doth therefore more exceed 180, than the other comes ſhort of it; and hath therefore the ſhorter Chord. So likewiſe, if N be 80; B is $160 = 180 - 20$; C, $240 = 180 + 60$: If N be 90, B is 180; C, $270 = 180 + 90$: If N be 100; B is $200 = 180 + 20 = 360 - 160$; C, $300 = 180 + 120 = 360 - 60$: Where the Triple is farther from a Semicircumference, as more exceeding it; and nearer to a whole Revolution (which is Equivalent to nothing) as approaching nearer to it; and therefore the Chord of the Triple, leſs than that of the double: So, if N, or L be 108; B is $216 = 180 + 36 = 360 - 144$; C, $324 = 360 - 36 = 180 + 144$. If $L = \frac{1}{5} + A$ be 120; B is $243 = 180 + 60 = 360 - 120$; C, 360. And therefore that B, the greater Chord: And ſo it will be 'till we come to 144 Degrees (or $\frac{4}{5}$) when again they will become equal; for then B will be $288 = 180 + 108 = 360 - 72$; and $C = 432 = 360 + 72 = 540 - 108$; which doth as much ſurpaſs a whole Revolution as the other wants of it; and doth as much want of a third Semicircumference as the other exceeds the firſt; and therefore their Chords become equal. But after this, the Chord of the Triple doth again become the greater: For if M the ſingle Arch be 145; B will be $290 = 180 + 110 = 360 - 70$; C, $435 = 360 + 75 = 540 - 105$: If M be 150; B is $300 = 360 - 60$; C, $450 = 360 + 90$: If M be 180; B is 360; C, is $540 = 360 + 180$: If M be 200; B is $400 = 360 + 40$; C, $600 = 360 + 240$; where the Arch C (as farther remote from an intire Revolution) requires the greater Chord. And ſo onward 'till we come to 216, (or $\frac{3}{5}$) where the Chords of B and C do again become equal, for B will be $432 = 360 + 72$; C, $648 = 720 - 72$; where the Arch B doth as much exceed one Revolution, as C wants of two; and therefore require equal Chords. After this, the Arches L, N, from 216 to 288, have the ſame Chords with thoſe of L, N, from 144 backward to 72, (as being their Complements to a whole Revolution,) and the ſame Chords of their Doubles and Triples, with the Doubles and Triples of thoſe; and therefore (as there) the Chords of the Duple greater than thoſe of the Triple. And from thence to 360 (which is an entire Revolution) the Chords are the ſame with thoſe of A and E, (as being the Remainders of theſe to an intire Revolution) and therefore here alſo, the Chord of the Triple is greater than that of the Duple.

XCIV. All which depends on this General Conſideration; (which equally ſerves for all ſuch Comparings of Arches and their Subtenſes; and is therefore to be taken notice of, once for all.) That is,

XCV. *Arches equally diſtant from the beginning or end of* (one or more) *entire Revolutions, have equal Subtenſes*, (for the ſame Chord doth indifferently ſubtend both or all of them;) *But thoſe which are leſs diſtant from ſuch beginning or end, have the leſſer ſubtenſes*; (as neareſt approaching to nothing.)

XCVI. Again,

XCVI. Again, *Arches equally diſtant* (whether in Exceſs or defect) *from* 1, 2, 3, (or any odd number of) *Semicircumferences, have equal ſubtenſes*, (for here alſo the ſame Chord ſubtends both or all;) *but thoſe which are leſs diſtant from ſuch Semicircumferences, have the greater Subtenſe*, (as neareſt approaching to that of a Semicircumference, or 180 Degrees, the greateſt Chord of all.) *Fig.* XXIX

XCVII. 'Tis manifeſt therefore, that, if the Arch E or A be not greater than 60 Degrees, and conſequently the Triple Arch do not exceed one Semicircumference, That of the Treble (as neareſt approaching to it) will be greater than that of the Double. And though A be greater than 60 Degrees; that of the Triple will yet be the greater, 'till this do as much exceed a Semicircumference as the Double comes ſhort of it: That is, 'till $2\frac{1}{2}A = 180$ Deg. or $\frac{1}{2}$ of the whole Circumference; that is, 'till $A = \frac{1}{5}$, or 72 Degrees. And what is ſaid of E and A leſs than $\frac{1}{5}$, doth equally hold of $\frac{2}{5} + A = 1 - E$, and $\frac{2}{5} + E = 1 - A$, which have the ſame Chords with E and A; and their Double and Treble, the ſame with the Double and Treble of E and A.

XCVIII. But if N, or L, the ſingle Arch exceed $\frac{1}{5}$, ſuppoſe $\frac{1}{5} + E$ or $\frac{1}{5} + A$; the Subtenſe of the double will be the longer. For the Subtenſe of , being the ſame with that of $\frac{1}{5} = 1 - \frac{4}{5}$; that of $2N = \frac{2}{5} + 2E$ will be longer than it, as nearer approaching to $\frac{1}{2}$; ('till $2N$ or $2L = \frac{1}{2}$, that is, N or $L = \frac{1}{4}$ or 90 Degrees;) but that of $3N = \frac{3}{5} + 3E$ leſs than it, as nearer approaching to 1 intire Revolution. And even when 2 L exceeds $\frac{1}{2}$, yet 3 L will have the leſs Chord, as nearer approaching to 1 intire Revolution; 'till it become equal to it; that is, $3L = 1$, and $L = \frac{1}{3}$, or 120 Deg. And even after this, 'till 3 L do as much exceed 1, as 2 L comes ſhort of it; that is, 'till $2\frac{1}{2}L = 1$; or $L = \frac{2}{5}$ or 144 Degrees. But then (as before at A or $N = \frac{1}{5}$) the Chords will be equal; for then the double is $\frac{4}{5} = 1 - \frac{1}{5}$; the Treble $\frac{6}{5} = 1 + \frac{1}{5}$. And what is ſaid of $N = \frac{1}{5} + E$, or $L = \frac{1}{5} + A$; holds equally true of $N = \frac{1}{5} + A$, or $L = \frac{1}{5} + E$; (that is of $1 - N$, or $1 - L$;) as having the ſame Chords with thoſe.

XCIX. But when M the ſingle Arch exceeds , ſuppoſe $\frac{2}{5} + E$; the Chord of the Treble will again be longer than that of the Double. For the Treble of $\frac{2}{5}$ as much exceeding, as the Double of it comes ſhort of, 1 Revolution; the Treble of $\frac{2}{5} + E$ will more exceed it, (approaching nearer to the third Semicircumference) and the Double want leſs of it, (approaching nearer to 1 Revolution,) 'till $3M = \frac{2}{2}$; that is, $M = \frac{1}{2}$ or 180 Degrees. And what is ſaid of $M = \frac{2}{5} + E$ leſs than $\frac{1}{2}$; holds alſo of $M = \frac{2}{5} + A = \frac{4}{5} - E$: Which doth as much exceed a Semicircumference, as the other comes ſhort of it.

C. 'Tis manifeſt therefore, that for the Arches A, E, leſs than $\frac{1}{5}$, or more than $\frac{4}{5}$ (but leſs than 1 Revolution;) and again for the Arch M, more than $\frac{2}{5}$ but leſs than $\frac{1}{5}$; the Chord of the Triple is greater than that of the Double; by § 97, 99. But, for the Arches L or M, more than $\frac{1}{5}$ but leſs than $\frac{2}{5}$; or more than $\frac{4}{5}$, but leſs than $\frac{4}{5}$; the Chord of the Double is greater than that of the Treble; by § 98. But in caſe the ſingle Arch be $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$; (or any number of Quintants,) the Chords of the Double and Treble are equal. And the ſame method may be purſued in other like Compariſons of Arches and Chords.

CI. Now, (to return where we left off at § 92.) what hath been particularly delivered, may be Collected into this General. Namely, (putting O for the Subtenſe of the ſingle Arch) $Cq \backsim Bq = OF$ (by § 85, 88, 92.) And $\frac{Cq \backsim Bq}{O} = F$. And $\frac{Cq \backsim Bq}{F} = O$. That is,

Fig. XXIX. CII. *The Difference of the Squares of the ſubtenſes of the Triple and double Arches, is equal to the Rect-angle of the ſubtenſes of the ſingle and Quintuple. And that Difference applied to either of theſe, gives the other.* (Which is a General to that of § 3.) Namely, if O be interpreted of A, E, M; then $Cq - Bq = OF$, and $\frac{Cq - Eq}{O} = F$. If, of L, N; then $Bq - Cq = OF$, and $\frac{Bq - Cq}{O} = F$. If, of V, X; then $Bq \backsim Cq = VF = o$. And $\frac{Bq \backsim Cq}{V} = F = o$. Or, $Bq \backsim Cq = XF = o$, and $\frac{Bq \backsim Cq}{X} = F = o$.

CIII. Again, becauſe $Cq \backsim Bq = C + B$ into $C \backsim B$; therefore, $O . B + C :: B \backsim C$. (interpreting $C \backsim B$, of $C - B$, for A, E, M; but of $B - C$, for L, N.) That is,

CIV. *As the Subtenſe of the ſingle Arch, to the Aggregate of the ſubtenſes of the Double and Triple; ſo is the Difference of theſe, to that of the Quintuple.* (Which is a General to that of § 6.)

CV. But (by § 45, *Chap.* 30.) $C = 3O \backsim \frac{Oc}{Rq}$; and therefore, $Cq = 9Oq - \frac{6Oqq}{Rq} + \frac{Occ}{Rqq}$: And (by § 7, *Chap.* 29.) $B = \sqrt{:} 4Oq - \frac{Oqq}{Rq}$; and therefore $Bq = 4Oq - \frac{Oqq}{Rq}$: From hence therefore, we may have the value of $Cq \backsim Bq = OF$, and of $\frac{Cq \backsim Bq}{O} = F$, ſutable to each caſe. Namely,

CVI. If the Arch O be leſs than $\frac{1}{5}$, or more than $\frac{4}{5}$; (that is, from o, to 72° and from 288, to 360.) Or more than $\frac{2}{5}$ but leſs than $\frac{3}{5}$, (that is, from 144, to 216°) there is $OF = Cq - Bq = 9Oq - \frac{6Oqq}{Rq} + \frac{Occ}{Rqq} - 4Oq + \frac{Oqq}{Rq} = 5Oq - \frac{5Oqq}{Rq} + \frac{Occ}{Rqq}$: And $F = 5O - \frac{5Oc}{Rq} + \frac{Ocq}{Rqq}$. And O to be underſtood of A, E, and M.

CVII. But if the Arch O be more than $\frac{1}{5}$ but leſs than $\frac{2}{5}$; or more than $\frac{3}{5}$ but leſs than $\frac{4}{5}$; (that is, from 72 Degrees to 144, and from 216 to 288:) Then is, $OF = Bq - Cq = 4Oq - \frac{Oqq}{Rq} - 9Oq + \frac{6Oqq}{Rq} - \frac{Occ}{Rqq} = -5Oq + \frac{5Oqq}{Rq} - \frac{Occ}{Rqq}$: And $F = -5O + \frac{5Oc}{Rq} - \frac{Ocq}{Rqq}$. And O to be interpreted of L, N.

CVIII. That is, (to reduce all to a brief Synopſis)

From	0° to 36.°	$5E - \frac{5Ec}{Rq} + \frac{Ecq}{Rqq} = F.$	From	324° to 360.°
	36 ... 72.	$5A - \frac{5Ac}{Rq} + \frac{Acq}{Rqq} = F.$		288 ... 324.
	72 ... 108.	$-5N + \frac{5Nc}{Rq} - \frac{Ncq}{Rqq} = F.$		252 ... 288.
	108 ... 144.	$-5L + \frac{5Lc}{Rq} - \frac{Lcq}{Rqq} = F.$		216 ... 252.
From	144 to 180.	$+5M - \frac{5Mc}{Rq} + \frac{Mcq}{Rqq} = F.$	From	180 to 216.

And, in the common term (or Point of connexion) of theſe Intervals, it is indifferent to whether of the two to refer them: As at 36 Degrees, to E or A; at 72° to A or N; and ſo of the reſt.

CIX. Hence

CIX. Hence follows this Five-fold Equation; containing five Roots. $5RqqE - 5RqEc + Eqc = 5RqqA - 5RqAc + Aqc = (RqqF=) - 5RqqN + 5RqNc - Nqc = -5RqqL + 5RqLc - Lqc = +5RqqM - 5RqMc + Mqc$. *Fig.* XXIX.

CX. Now becauſe $5RqqA - 5RqAc + Aqc = 5RqqE - 5RqEc + Eqc$; therefore, (by tranſpoſition) $5RqqA - 5RqqE = 5RqAc - 5RqEc - Aqc + Eqc$: And (dividing all by $A - E$,) $5Rqq = \frac{Ac - Ec}{A - E} 5Rq - \frac{AqC - Eqc}{A - E}$.

CXI. But (by § 13, 14, 15, *Chap.* 30.) $\frac{Ac - Ec}{A - E} = Aq + AE + Eq = 3Rq$, and therefore, $\frac{Ac - Ec}{A - E} 5Rq = 15Rqq$: Therefore, $5Rqq = 15Rqq - \frac{Aqc - Eqc}{A - E}$: That is, $\frac{Aqc - Eqc}{A - E} = 10Rqq$.

CXII. And again, becauſe (as will appear by Diviſion) $\frac{Aqc - Eqc}{A - E} = Aqq + AcE + AqEq + AEc + Eqq$: Therefore, $Aqq + AcE + AqEq + AEc + Eqq = 10Rqq$.

CXIII. But the Angle contained by A, E, is of 144 Degrees. (As being an Angle in the Circumference inſiſting on an Arch of 288 Degrees, or $\frac{4}{5}$ of the whole.) Therefore,

CXIV. *The Difference of the Quadricubes of the Legs containing an Angle of* 144 *Degrees, or divided by the Difference of thoſe Legs, is equal to Ten Biquadrates of the Radius of the Circumſcribed Circles.* (by § 111.) That is, (by § 112.)

CXV. *The Biquadrates of the Legs containing an Angle of* 144 *Degrees, together with the three means proportional between theſe Biquadrates, is equal to Ten Biquadrates of the Radius of the Circumſcribed Circle.*

CXVI. But now the Baſe of this Triangle, being the ſide of an Inſcribed Equilater Pentagon, or Subtenſe of 72 Degrees; is, (by § 31.) $R\sqrt{\frac{5 - \sqrt{5}}{2}}$; and therefore the Square of this $\frac{5 - \sqrt{5}}{2} Rq$; and, its Biquadrate, $\frac{30 - 10\sqrt{5}}{4} Rqq = \frac{15 - 5\sqrt{5}}{2} Rqq$: Which is, to $10Rqq$, as $\frac{15 - 5\sqrt{5}}{2}$ to 10; or as $3 - \sqrt{5}$ to 4. Therefore,

CXVII. *The Difference of the Quadricubes of the Legs containing an Angle of* 144 *Degrees, divided by the Difference of thoſe Legs, or the Biquadrates of the Legs containing ſuch Angle, together with the three means Proportional between theſe Biquadrates; is, to the Biquadrate of the Baſe ſubtending that Angle; as* 4 *to* $3 - \sqrt{5}$.

CXVIII. Again, becauſe (by § 109.) $5RqqM - 5RqMc + Mqc = (RqqF=) 5RqqA - 5RqAc + Aqc$ (M being greater than A;) therefore (as at § 110, 111, 112,) $5Rqq = \frac{Mc - Ac}{M - A} 5Rq - \frac{Mqc - Aqc}{M - A} = 15Rqq - \frac{Mqc - Aqc}{M - A}$. And $\frac{Mqc - Aqc}{M - A} = 10Rqq = Mqq + McA + MqAq + MAc + Aqq$.

Fig. XXIX. CXIX. And (by the ſame reaſon) $5Rqq = \frac{Mc-Ec}{M-E}5Rq - \frac{Mqc-Eqc}{M-E} = 15Rqq - \frac{Mqc-Eqc}{M-E}$. And $\frac{Mqc-Eqc}{M-E} = 10Rqq = Mqq + McE + MqEq + MEc + Eqq$.

CXX. And alſo, becauſe (by § 109.) $-5RqqL + 5RqLc - Lqq = (RqqF =) -5RqqN + 5RqNc - Nqc$; and (changing all the ſigns) $5RqqL - 5RqLc + Lqc = 5RqqN - 5RqNc + Nqc$ (L being greater than N:) Therefore, (as at § 118.) $5Rqq = \frac{Lc-Nc}{L-N}5Rq - \frac{Lqc-Nqc}{L-N} = 15Rqq - \frac{Lqc-Nqc}{L-N}$. And $\frac{Lqc-Nqc}{L-N} = 10Rqq = Lqq + LcN + LqNq + LNc + Nqq$.

CXXI. But the Angles contained by M, A; and M, E; and L, N; are of 72 Degrees; (as being Angles in the Circumference inſiſting on an Arch of 144 Degrees, or $\frac{2}{5}$ of the whole:) And, as to M, E; one of the Angles at the Baſe, obtuſe; but, as to M, A; all acute: (This being an Angle in a greater Segment; that, in a leſs, than a Semicircle;) and likewiſe, as to L, N, all acute. Therefore, (as at § 114, 115.)

CXXII. *The Difference of the Quadricubes of the Legs containing an Angle of* 72 *Degrees, divided by the Difference of thoſe Legs; is equal to Ten Biquadrates of the Radius of the Circumſcribed Circle.* And,

CXXIII. *The Biquadrates of the Legs containing an Angle of* 72 *Degrees, together with the three means Proportional between thoſe Biquadrates, is equal to Ten Biquadrates of the Radius of the Circumſcribed Circle.*

CXXIV. But here the Baſe of this Triangle (ſubtended to that Angle of 72 Degrees,) is the Subtenſe of a Biquintant, or Triquintant; that is, of $\frac{2}{5} = 144$ Degrees; or of $\frac{3}{5} = 216$ Degrees; which is (by § 54.) $R\sqrt{\frac{5+\sqrt{5}}{2}}$. And the Square of this $\frac{5+\sqrt{5}}{2}Rq$: And its Biquadrate $\frac{30+10\sqrt{5}}{4}Rqq = \frac{15+5\sqrt{5}}{2}Rqq$. Which is, to $10Rqq$, as $\frac{15+5\sqrt{5}}{2}$ to 10; or, as $3+\sqrt{5}$ to 4. Therefore,

CXXV. *The Difference of the Quadricubes of the Legs containing an Angle of* 72 *Degrees divided by the Difference of thoſe Legs*; or, *the Biquadrates of the Legs containing ſuch Angle, together with the three means Proportional between thoſe Biquadrates; is, to the Biquadrate of the Baſe ſubtending that Angle; as* 4 *to* $3+\sqrt{5}$.

CXXVI. Again, becauſe (by § 109,) $5RqqA - 5RqAc + Aqc = (RqqF =) -5RqqL + 5RqLc - Lqc$ (L being greater than A:) Therefore, (by tranſpoſition) $5RqqL + 5RqqA = 5RqLc + 5RqAc - Lqc - Aqc$. And (dividing all by $L+A$,) $5Rqq = \frac{Lc+Ac}{L+A}5Rq - \frac{Lqc+Aqc}{L+A}$.

CXXVII. But (by § 46, 47, *Chap.* 30.) $\frac{Lc+Ac}{L+A} = Lq - LA + Aq = 3Rq$, and therefore, $\frac{Lc+Ac}{L+A}5Rq = 15Rqq$: Therefore, $5Rqq = 15Rqq - \frac{Lqc+Aqc}{L+A}$. That is, $\frac{Lqc+Aqc}{L+A} = 10Rqq$.

CXXVIII. And

CXXVIII. And again, becauſe (as will appear by Diviſion) $\frac{Lqc+Aqc}{L+A} = Lqq - LcA + LqAq - LAc + Aqq$: Therefore, $Lqq - LcA + LqAq - LAc + Aqq = 10Rqq$. *Fig.* XXIX.

CXXIX. But the Angle contained by L, A, is of 36 Degrees (as being an Angle at the Circumfererence inſiſting on an Arch of 72 Degrees, or $\frac{1}{5}$ of the whole,) and one of the other, obtuſe.

CXXX. And the ſame is to be ſaid (for the ſame reaſons) of N, E, as of L, A.

CXXXI. And alſo, becauſe in like manner (by § 109,) $5RqqM - 5RqMc + Mqc = (RqqF =) - 5RqqN + 5RqNc - Nqc$; (M being greater than N:) Therefore, (by the ſame methods,) $\frac{Mqc+Nqc}{M+N} = 10Rqq = Mqq - McN + MqNq - MNc + Nqc$. And the Angle contained by M, N, is of 36 Degrees; and one of the other, obtuſe.

CXXXII. And juſt the ſame (for the ſame reaſons) of M, L; ſave that here the Angles be all acute.

CXXXIII. And theſe are all the caſes that can happen, the Angle at the Vertex being 36 Degrees; for that of the Legs V, X; is to be reduced to that of A, L; and that of X, X; to that of L, M; (and the like is to be underſtood of other like caſes, where A is extended to the whole Quintant, and E vaniſheth into nothing.) Therefore,

CXXXIV. *The Sum of the Quadricubes of the Legs containing an Angle of* 36 *Degrees, divided by the Sum of thoſe Legs, is equal to Ten Biquadrates of the Radius of the Circumſcribed Circle.* (By § 127, 130, 131, 132.) And,

CXXXV. *The Biquadrates of the Legs containing an Angle of* 36 *Degrees, with a mean Proportional between thoſe Biquadrates, wanting the firſt and third of three means Proportional between them; are equal to Ten Biquadrates of the Radius of the Circumſcribed Circle.*

CXXXVI. But the Baſe ſubtended to this Angle of 36 Degrees, being the ſide of an inſcribed Equilater Pentagon; (as at § 116,) the Biquadrate hereof is to 10 Rqq as $3 - \sqrt{5}$ to 4. And therefore,

CXXXVII. *The Sum of the Quadricubes of the Legs containing an Angle of* 36 *Degrees, divided by the Sum of thoſe Legs:* Or, *the Biquadrates of the Legs containing ſuch Angle, with a mean Proportional between thoſe Biquadrates, wanting the firſt and third of three mean Proportionals between them; is, to the Biquadrate of the Baſe ſubtending that Angle; as* 4, *to* $3 - \sqrt{5}$.

CXXXVIII. Again, becauſe (by § 109.) $5RqqA - 5RqAc + Aqc = (RqqF =) - 5RqqN + 5RqNc - Nqc$; (N being greater than A;) Therefore, (as at § 126, *&c.*) $\frac{Nqc+Aqc}{N+A} = 10Rqq = Nqq - NcA + NqAq - NAc + Aqq$.

CXXXIX. And, in like manner, becauſe $5RqqE - 5RqEc + Eqc = (RqqF =) - 5RqqL + 5RqLc - Lqc$, (L being greater than E:) Therefore, $\frac{Lqc+Eqc}{L+E} = 10Rqq = Lqq - LcE + LqEq - LEc + Eqq$.

CXL. But

Fig. XXIX. CXL. But the Angles contained by N, A; or by L E; are Angles of 108 Degrees, (as being Angles at the Circumference, inſiſting on an Arch of 216 Degrees, or $\frac{1}{5}$ of the whole.) Therefore,

CXLI. *The Sum of the Quadricubes of the Legs containing an Angle of* 108 *Degrees, divided by the Sum of thoſe Legs, is equal to Ten Biquadrates of the Radius of a Circumſcribed Circle.* And,

CXLII. *The Biquadrates of the Legs containing an Angle of* 108 *Degrees, with a mean Proportional between thoſe Biquadrates, wanting the firſt and third of three means Proportional between them; are equal to Ten Biquadrates of the Radius of a Circumſcribed Circle.*

CXLIII. But the Baſe ſubtended to this Angle of 108 Degrees, is the Subtenſe of a Biquintant, or (which is the ſame) of a Triquintant; that is, of $\frac{2}{5}$ or $\frac{3}{5}$ of the whole Circumference: And therefore, (as at § 124) is to 10 Rqq, as $3+\sqrt{5}$ to 4. Therefore,

CXLIV. *The Sum of the Quadricubes of Legs containing an Angle of* 108 *Degrees, divided by the Sum of thoſe Legs:* Or, *The Biquadrates of the Legs containing ſuch Angle, with a mean Proportional between thoſe Biquadrates, wanting the firſt and third of three means Proportional between them; is, to the Biquadrate of the Baſe ſubtending that Angle; as* 4 *to* $3+\sqrt{5}$.

CXLV. Now theſe ſeveral Theorems thus delivered in particular, may be Collected into theſe Generals following. Namely,

CXLVI. *The Difference of the Quadricubes of Legs containing an Angle of* 144 *or of* 72 *Degrees, divided by the Differences of thoſe Legs:* Or, *The Sum of the Quadricubes of Legs containing an Angle of* 36 *Degrees, or of* 108 *Degrees, divided by the Sum of thoſe Legs:* Or, (which is Equivalent to thoſe) *The Biquadrates of the Legs* (in the former caſe) *with the three means Proportional between them;* Or, (in the latter caſe) *The Biquadrates of the Legs, with a mean Proportional between them, wanting the firſt and third of three means Proportionals:* Are equal to *Ten Biquadrates of the Radius of a Circumſcribed Circle.* And, *Theſe, to the Biquadrates of their reſpective Baſes ſubtending ſuch Angle of* 144 *Degrees, or of* 36 *Degrees;* are as 4 to $3-\sqrt{5}$; *but, of the Baſes ſubtending ſuch Angle of* 72 *Degrees, or of* 108 *Degrees; as* 4 *to* $3+\sqrt{5}$: Or, as 8 to $6-2\sqrt{5}$, and 8 to $6+2\sqrt{5}$. That is, in the Duplicate proportion $2\sqrt{2}$ to $\sqrt{5}-1=\sqrt{:6-2\sqrt{5}}$; and of $2\sqrt{2}$ to $\sqrt{5}+1=\sqrt{:6+2\sqrt{5}}$.

CXLVII. And thoſe ſides, contain theſe following Angles.

Sides.	Deg.	Sides.	Deg.	Sides.	Deg.
A, E.	144	L, A.	36	N, A.	108.
M, A.	72	N, E.		L, E.	
M, E.		M, L.			
L, N.		M, N.			

whereof the Four firſt Couple, are ſides of like ſigns; the ſix latter, of unlike.

CXLVIII. The ſame Equations may be thus alſo conſidered. Becauſe (by § 110) $5RqqA-5RqqE=5RqAc-5RqEc-Aqc+Eqc$: Therefore, (dividing all by $A-E$, and again by $5Rq$,) $Rq=Aq+AE+Eq-\frac{Aqq+AcE+AqEq+AEc+Eqq}{5Rq}$ And (by tranſpoſition)

$Aq+AE+Eq-Rq$, into $5Rq$, $=Aqq+AcE+AqEq+AEc+Eqq$.

CXLIX. And

CXLIX. And in like manner (becauſe M, A, and M, E, and L, N, have alſo like ſigns.) *Fig.* XXIX.

Mq-|-MA-|-Aq—Rq, into 5Rq, = Mqq-|-McA-|-MqAq-|-MAc-|-Aqq.

Mq-|-ME-|-Eq—Rq, into 5Rq, = Mqq-|-McE-|-MqEq-|-MEc-|-Eqq.

Lq-|-LN-|-Nq—Rq, into 5Rq, =Lqq-|-LcN-|-LqNq-|-LNc-|-Nqq.

CL. And therefore, *In a Right-lined Triangle, whoſe Legs contain an Angle of* 144 *Degrees*, (as A, E,) *or* 72 *Degrees*, (as M, A, or M, E, or L, N,) *if the Squares of the Legs, with the Rectangle of them, wanting the Square of the Radius of the Circumſcribed Circle, be all Multiplied into five times the Square of that Radius: The Product is equal to the Biquadrates of the Legs, with three means Proportional between thoſe Biquadrates.*

CLI. In like manner may be ſhewed, (where the ſigns of the Legs be unlike,) That,

Lq—LA-|-Aq—Rq, into 5Rq, = Lqq—LcA-|-LqAq—LAc-|-Aqq.

Nq—NE-|-Eq—Rq, into 5Rq, = Nqq—NcE-|-NqEq—NEc-|-Eqq.

Mq—ML-|-Lq—Rq, into 5Rq, = Mqq—McL-|-MqLq—MLc-|-Lqq.

Mq—MN-|-Nq—Rq, into 5Rq, =Mqq—McN-|-MqNq—MNc-|-Nqq.

Nq—NA-|-Aq—Rq, into 5Rq, = Nqq—NcA-|-Nq Aq—NAc-|-Aqq.

Lq—LE-|-Eq—Rq, into 5Rq, = Lqq—Lc E-|-Lq Eq—LE c-|-Eqq.

CLII. And therefore, *In a Right-lined Triangle, whoſe Legs contain an Angle of* 36 *Degrees*, (as L, A, or N, E, or M, L, or M, N;) *or* 108 *Degrees*, (as N, A, or L, E;) *if the Squares of the Legs; wanting the Rectangle of them, and the Square of the Radius of the Circumſcribed Circle, be all Multiplied into five times the Square of that Radius; the Product is equal to the Biquadrates of the Legs, and a mean Proportional between thoſe Biquadrates, wanting the firſt and third of three means Proportional between them.*

CLIII. Now all this variety of caſes, and Deductions from them; from § 83, hitherto, ariſeth from the firſt Conſtruction, at § 1. and what is Analogous thereunto: Where the ſix Lines, for the four ſides and two Diagonals of the Quadrilater, are F, A; B, B; C, C. And the variety ariſeth from hence, that ſometimes C, C, are the Diagonals; and B, B, oppoſite ſides; ſometimes C, C, are oppoſite ſides; and B, B, Diagonals; according as C or B happens to be greater. *Fig.* XXIV.

CLIV. But, by a like method, with ſome little alteration, we may infer moſt of the ſame things; (and obſerve thence like Deductions, or others Analogous thereunto; with like variety of caſes;) from the ſecond Conſtruction, at § 9; where the ſix Lines are, F, C; A, A; D, D. Where the variety of caſes proceedeth from hence, that ſometimes D, D, are Diagonals, and A, A, oppoſite ſides; ſometimes D, D, are oppoſite ſides; and A, A, (or what anſwers to them) Diagonals; according as D, or A, (or what anſwers to this, E, L, M, N,) are greater. *Fig.* XXV.

CLV. And accordingly, the propoſitions at § 10, and 12, may be delivered more generally. Namely,

CLVI. *The*

Fig. XXV. CLVI. *The Difference of the Squares of the ſubtenſes of the Quadruple and of the ſingle Arch; is equal to the Rectangle of the ſubtenſes of the Triple and Quintuple.* And, being divided by either of theſe, it gives the other. And,

CLVII. *As the Subtenſe of the Triple Arch, to the Sum of the ſubtenſes of the Quadruple and of the ſingle; ſo is the Differences of theſe, to the Subtenſe of the Quintuple.* Whether ſuch ſingle Arch be leſſer, or greater, or equal to a Quintant.

Fig. XXVI. CLVIII. And in like manner, from the Third Conſtruction, at § 14. where the ſix Lines are, F, A; C, A; B, D. And what is there delivered (at § 15, 17.) of an Arch leſs than a Quintant, may be more generally delivered, thus,

CLIX. *The Difference of the Rectangles of the ſubtenſes of the double and of the Quadruple Arch; and, of the ſingle and Triple; is equal to that of the Subtenſes of the ſingle and Quintuple.* And, being divided by either of theſe, it gives the other. And,

CLX. *As the Subtenſe of the ſingle Arch, to that of the double; ſo is that of the Quadruple, to the Sum or Difference of the ſubtenſes of the Triple and Quintuple; according as* B, D, *happen to be Diagonals or oppoſite ſides*

Fig. XXVII. CLXI. And in like manner, from the Fourth Conſtruction, at § 20; where the ſix Lines are, F, B; A, B; C, D. And what is there delivered at § 21, 23, may be more generally delivered; thus,

CLXII. *The Difference of the Rectangles, of the ſubtenſes of the Triple and Quadruple Arches; and, of the ſingle and double; is equal to that of the ſubtenſes of the double and Quintuple.* And, being divided by either, gives the other of them. And,

CLXIII. *As the Subtenſe of the double Arch, is to that of the Triple; ſo is that of the Quadruple, to the Sum or Difference of the ſubtenſes of the Quintuple, and ſingle; according as* C, D, *happen to be Diagonals, or oppoſite ſides.*

CLXIV. And from every of theſe Conſtructions, may be derived like varietie of caſes and Conſequences, (with Figures ſuited to thoſe caſes:) as (at § 83, *&c.*) is done from the firſt Conſtruction. But I forbear to purſue theſe any farther; and leave it to any who ſhall think fit, (for their own Exerciſe,) to purſue theſe as I have done the firſt.

CHAP.

CHAP. V.

Of the Sextuplation, and Sextiſection of an ARCH *or* ANGLE: *And other following Multiplications and Sections.*

I. ACCORDING to the ſame methods may be had, the Sextuplation, Septuplation, and other conſequent Multiplications; as alſo the Sextiſection, Septiſection, and other conſequent Sections, of an Arch or Angle. Of which I ſhall briefly touch at ſome.

II. The Sextuplation, may be had, by Tripling the Double, or Doubling the Triple Arch. And, accordingly, the Sextiſection, by Biſecting the Subtriple, or Triſecting the Subduple. (as is of it ſelf manifeſt.) And the ſame holds, in like manner, for Multiplications and Sections which take their Denomination from a Compound number. For Multiplications and Sections ſucceſſively made, according to the Components of ſuch Compound number, amount to the ſame as one by ſuch Compound number.

III. But though *Six* were not a Compound number, or be not conſidered as ſuch; yet may ſuch Sextuplation and Sextiſection be had in like manner as thoſe before. Namely,

IV. If in a Circle be inſcribed a Quadrilater, whoſe oppoſite ſides are B, B, ſubtenſes of the Duple; and B, G, ſubtenſes of the Duple and Sextuple; and the Diagonals D, D, ſubtenſes of the Quadruple. Then is, $Dq - Bq = BG$; and $B) Dq - Bq (G$. *Fig.* XXXIII

V. Or, Let the oppoſite ſides be A, A, and D, G; and the Diagonals F, F. Then is, $Fq - Aq = DG$; and $D) Fq - Aq (G$. *Fig.* XXXIV.

VI. Or, Let the oppoſite ſides be A, B, and C, G; and the Diagonals D, F. Then is, $DF - AB = CG$; and $C) DF - AB (G$. *Fig.* XXXV.

VII. Or, Let the oppoſite ſides be A, C, and B, G; the Diagonals C, F. Then $CF - AC = BG$; and $B) CF - AC (G$. *Fig.* XXXVI.

VIII. And therefore, $Dq - Bq = CF - CA$.

IX. Or, Let the oppoſite ſides be A, G, and A, D; the Diagonals B, F. Then $BF - AD = AG$; and $A) BF - AD (G$. *Fig.* XXXVII

X. Or, Let the oppoſite ſides be B, C, and A, G; the Diagonals C, D. Then $CD - BC = AG$; and $A) CD - BC (G$. *Fig.* XXXViii

XI. And therefore, $BF - AD = CD - BC$.

XII. It is manifeſt that from hence may be deduced a great number of Equations, and Analogies, and great variety of Theorems, in like manner, as is done in the Chapters foregoing. But I forbear here to purſue them in particular as is there done.

Fig. XXXVIII. XIII. But from every of thoſe Conſtructions, (the values of B, C, D, F, being known as is above declared,) we have (by ordering the Equations in due manner,) $G = \frac{12RccA - 19RqqAc + 8RqAqc - Aqqc}{Rqc\surd : 4Rq - Aq :}$. Or, $GRqc\surd : 4Rq - Aq : = 12RccA - 19RqqAc + 8RqAqc - Aqqc$. And (taking the Squares of theſe,) $4GqRcccc - GqRqqcc Aq = 144Rcccc Aq - 456RqqccAqq + 553RqccAcc - 328RccAqcc + 102RqqAqqcc - 16RqAcccc + Aqcccc$.

XIV. That is, (dividing all by $4Rq - Aq$,) $RqqccGq = 36RqqccAq - 105RqccAqq + 112RccAcc - 54RqqAqcc + 12RqAqqcc - Acccc$.

Fig. XXXIX. XV. Of this Equation there be Six plain Roots, anſwering to Aq; the Square Roots of which, are A. Which are ſo many ſtreight Lines from ſome one Point of the Circumference, to the Six Angles of an inſcribed regular Hexagon. (So that, any one of them being known, the reſt are known alſo. And the like in all ſuch Equations.)

XVI. Of theſe, the Two leaſt, A, E, (which ſubtend, on the one ſide, to Arches leſs than a Sextant; and, on the other ſide to more than five Sextants;) And the Two greateſt, x, y, (which ſubtend to Arches greater than two Sextants, but leſs than four;) are Affirmative Roots; (becauſe the Subtendent of the double Arch is leſs than that of the Quadruple; and therefore $Dq - Bq$ an Affirmative Quantity:) But the Two betwen them I, K, (which ſubtend on the one ſide, to Arches greater than one Sextant but leſs than two; and on the other ſide, to Arches greater than four Sextants but leſs than five;) are Negatives, (becauſe of D leſs than B; and therefore $Dq - Bq$ a Negative Quantity;) G being in all, reputed Affirmative.

XVII. If a Chord be ſubtendent to juſt a Sextant, or two or more Sextants; it is indifferent to whether of the two caſes on either ſide it be referred; ſuppoſe $\frac{1}{6} = \frac{1}{6} \pm 0$. (which is to be underſtood in all caſes of like nature.) And when ever this happens, one of the Roots vaniſh, or become equal to nothing.

XVIII. For the Septuplation or Septiſection of an Arch or Angle; we ſhall have, according as the Quadrilater may be differently inſcribed, the Subtenſe of the Septuple Arch, $H = \frac{Dq - Cq}{A}$, or $\frac{Fq - Bq}{C}$, or $\frac{Gq - Aq}{F}$, or $\frac{GB - FA}{A}$, or $\frac{Gc - DA}{B}$, or $\frac{GD - CA}{C}$, or $\frac{GF - BA}{D}$, or $\frac{FC - DE}{A}$, or $\frac{FD - CB}{B}$.

XIX. From every of which Equations, (having the values of B, C, D, F, G, known as before,) we ſhall have (by due ordering ſuch Equation) $H = 7A - \frac{14Ac}{Rq} + \frac{7Aqc}{Rqq} - \frac{Aqqc}{Rcc}$: Or, $RccH = 7RccA - 14RqqAc + 7RqAqc - Aqqc$.

XX. The

XX. The Seven Roots of this Equation; are, ſo many ſtreight Lines from ſome one Point of the Circumference, to the Seven Angles of an inſcribed Regular Heptagon.

XXI. Of theſe Roots (putting H Affirmative,) the two leaſt are Affirmative; the two next, are Negative; the two next to theſe, are again Affirmative; and, the greateſt Negative.

XXII. And after the ſame manner we may proceed as far as we pleaſe: Collecting the conſequent Multiplications and Sections, by the help of thoſe Antecedent.

XXIII. And all ſuch as are denominated by a Compound number (as $4=2\times2$, $6=2\times3$, $8=2\times4=2\times2\times2$, $9=3\times3$, *&c.*) may, with more convenience, (at leſt, as to the Section, if not as to the Multiplication alſo,) be performed by two or more operations, according to the Components of ſuch Compound number.

XXIV. But, both theſe, and thoſe which are Denominated from Prime numbers, (as 3, 5, 7, 11, *&c.*) may (by ſuch inſcription of Quadrilaters) be Reduced to ſuch Equations, as will contain as many Roots as is the number from which ſuch Multiplication or Section takes its Denomination.

XXV. And, of theſe, thoſe which are Denominated by an *Even* number, will afford Equations having *Plain* Roots; the Square Root of which Plains, are the ſubtenſes of the Arches.

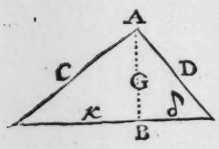
XXVI. But thoſe which are Denominated by *Odd* numbers, afford Equations whoſe Roots are thoſe ſubtenſes.

XXVII. And, of theſe ſubtenſes (as well in the one caſe as in the other,) the two leaſt (which I look upon as the Principal Roots of the Equation,) are Affirmatives (ſuppoſing the Subtenſe of the Multiple Arch to be always put Affirmative;) the two next greater than theſe, Negatives; the two next Affirmatives; and ſo onward, Alternately, as long as there be Roots remaining: ſave that, when the number is Odd, the greateſt of all will be ſingular, whereas the reſt go by Couples.

CHAP. VI.

Of the Proportion of the Baſe to the Legs of a Triangle, according as is the Angle at the Top of it.

I. THE noted Propoſition of *Pythagoras*, (which is in *Euclid*, 47 è 1.) concerning the Square of the Baſe equal to the Squares of the two Legs containing a Right-angle: And two more in *Euclid* (*Pr.* 12, 13, è 2.) concerning the Exceſs, (in caſe the Angle at the Top be Obtuſe;) or the Defect, (in caſe it be Acute;) of the Square of the Baſe, compared with the Squares of the two Legs: And ſome other Propoſitions in the foregoing Chapters, ſhewing what Proportion that Exceſs or Defect bears to a Rectangle of the Legs, in divers caſes: Gave me occaſion to purſue that Speculation a little further; according to the following Propoſitions.

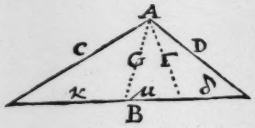


II. If by the Legs of a Triangle C, D, the Angle at the Top contained A, be a Right-angle, (or of 90 Degrees;) and from thence a Perpendicular G, on the Baſe, cut this into two Segments κ, δ: The two Triangles hence ariſing, κ G C, G δ D, are like to the whole C D B; (becauſe of one Angle common, the other a Right-angle, and therefore the third equal to the third.) And therefore (the Triangles being here deſigned by their ſides.)

$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \kappa \\ B \,.\, D :: D \,.\, \delta \end{array}\right\} \text{and therefore} \left\{\begin{array}{l} Cq = B\kappa \\ Dq = B\delta \end{array}\right\} \text{and therefore,}$$

$Cq + Dq = (B\kappa + B\delta = B$: into $\kappa + \delta = B :=) Bq$. That is,

The Square of the Baſe is equal to the two Squares of the Legs containing a Right-angle.



III. If the Angle A be 120 Deg. or $\frac{4}{3}$ of a Right-angle; and thence be drawn the Baſe two ſtreight Lines G, Γ; making, with it, Angles equal to that at the Top: The Two Triangles κ GC, Γ δ D, are like to the whole C D B, (becauſe of one Angle common, another equal to A, and therefore the third equal to the third;) with a Triangle between G Γ μ Equilater (becauſe each of the Angles at the Baſe, and therefore that at the Top, are of 60 Degrees:) And the Baſe $B = \kappa + \delta + \mu$. And therefore,

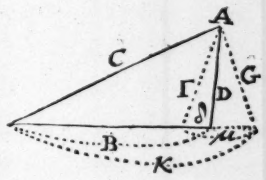
$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \kappa \\ B \,.\, D :: D \,.\, \delta \\ B \,.\, C :: D \,.\, \Gamma = G = \mu \end{array}\right\} \text{and} \left\{\begin{array}{l} Cq = B\kappa \\ Dq = B\delta \\ CD = B\mu \end{array}\right\} \text{Therefore,}$$

$Cq + Dq + CD = (B\kappa + B\delta + B\mu = B$: into $\kappa + \delta + \mu =) B :=$ B q. That is,

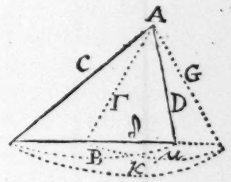
The Square of the Baſe (of an Angle of 120 Degrees) is equal to the Squares of the Legs and a Rectangle of them.

IV. If

IV. If the Angle A be 60 Degrees, or $\frac{2}{3}$ of a Right-angle; and from thence, G, Γ, making Angles with the Baſe equal to that of A: Then are, (for the ſame cauſes as before) $\varkappa$ G C, Γ δ D, like Triangles to C D B, (as before;) and G Γ μ an Equilater Triangle communicating with them: And the Baſe, B = $\varkappa$ + δ − μ. (or $\varkappa$ − μ + δ.) And therefore,



$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \varkappa \\ B \,.\, D :: D \,.\, \delta \\ B \,.\, C :: D \,.\, \Gamma = G = \mu \end{array}\right\} \text{and} \left\{\begin{array}{l} Cq = B\varkappa \\ Dq = B\delta \\ CD = B\mu \end{array}\right.$$

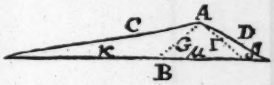


And therefore, Cq + Dq − CD = (B$\varkappa$ + Bδ − Bμ = B: into $\varkappa$ + δ − μ = B =) Bq. That is,

The Square of the Baſe (of an Angle of 60 Degrees) is equal to the Squares of the Legs, wanting the Rectangle of them.

(Note here, that, by $\varkappa$, I underſtand the Baſe to the Legs CG; by δ, that of the Legs D Γ; by μ, that of G Γ; and, by B, that of C D, which is ever equal to $\varkappa$ + δ ± μ, however theſe parts be intermingled. Which where it is + μ, is commonly more obvious to the Eye; but where it is − μ, is more perplex, and will need more conſideration to diſcern; but it is equally true in both caſes.)

V. If A be 135 Degrees (or $\frac{3}{8}$ of a Quadrant,) and G Γ drawn (as before) to make the like Angle with the Baſe: The Triangles $\varkappa$ G C, Γ δ D, will be like to C D B; and G Γ μ will be Equicrural, making the Angles at the Baſe, of 45 Degrees, (ſo much as A wants of two Right-angles;) and therefore, the Angle at the Vertex (which I ſhall call V,) of 90 Degrees. And therefore, (by § 2.) $\mu q = 2Gq$, and $\mu = G\sqrt{2}$. And the Baſe B = $\varkappa$ + δ + μ. And therefore,

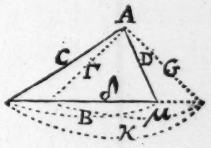


$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \varkappa \\ B \,.\, D :: D \,.\, \delta \\ B \,.\, C :: D \,.\, \Gamma = G \end{array}\right\} \text{and} \left\{\begin{array}{l} Cq = B\varkappa. \\ Dq = B\delta. \\ CD = BG. \\ CD\sqrt{2} = B\mu. \end{array}\right. \text{Therefore,}$$

Cq + Dq + CD$\sqrt{2}$ = (B$\varkappa$ + Bδ + Bμ = B: into $\varkappa$ + δ + μ = B: =) Bq. That is,

The Square of the Baſe (of an Angle of 135 Degrees) is equal to the Squares of the Legs, with a Rectangle of them Multiplied into $\sqrt{2}$.

VI. If A be 45 Degrees: It will in like manner be ſhewed, that (becauſe of B = $\varkappa$ + δ − μ.)



Cq + Dq − CD$\sqrt{2}$ = (B$\varkappa$ + Bδ − Bμ = B: into $\varkappa$ + δ − μ = B: =) Bq. That is,

The Square of the Baſe (of an Angle of 45 Degrees) is equal to the Squares of the Legs, wanting a Rectangle of them Multiplied into $\sqrt{2}$.

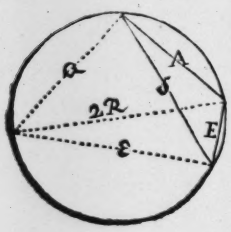
VII. And,

VII. And, univerſally, what ever be the Angle A; it will (by like proceſs) be ſhewed: That,

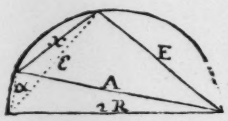
$$Bq = Cq + Dq \pm \frac{\mu}{G} CD.$$ That is,

The Square of the Baſe (whatever be the Angle at the Vertex) is equal to the Squares of the Legs, together with (if it be greater than a Right-angle) or wanting (if leſs than ſuch) a Plain, which ſhall be, to the Rect-angle of the Legs, as a Portion in the Baſe-line, intercepted between two Lines from the Vertex, making at the Baſe a like Angle with that of the Vertex, to one of thoſe two Lines ſo drawn.

VIII. Of this we are to give great variety of Examples in the following Chapter, where this General Theorem is applied to particular caſes: And which is further improved by theſe two enſuing Propoſitions.



IX. *The Radius of a Circle, with the ſubtenſes of two Arches, being given; the Subtenſe of their Aggregate is alſo given.* For, ſuppoſing the ſubtenſes of the given Arches to be A, E: The ſubtenſes of their Remainders to a Semicircle, are alſo had: Suppoſe $\alpha = \sqrt{:4Rq - Aq}$: And $\varepsilon = \sqrt{:4Rq - Eq}$. And therefore, inſcribing a Quadrilater whoſe oppoſite ſides are A, ε; and E, α; one of the Diagonals is the Diameter $= 2R$; the other the Subtenſe of the Sum or Aggregate of thoſe Arches, ſuppoſe $S = \frac{A\varepsilon + E\alpha}{2R}$.

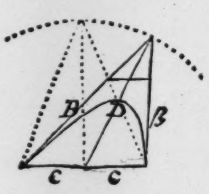


X. *The ſame being given; the Subtenſe of the Difference of thoſe Arches is alſo given.* For, having (as before) A, α; E, ε; 2R: we have (by a Quadrilater duly inſcribed) the Subtenſe of the Difference, $X = \frac{A\varepsilon - E\alpha}{2R}$.

XI. It is manifeſt alſo, (from what is before delivered,) that the ſame Triangle $G\Gamma\mu$, doth indifferently ſerve for the Angle of 120 Degrees and of 60 Degrees: And, in like manner, for 135, and 45: And ſo, for any two Arches whereof one doth as much exceed as the other wants of a Quadrant. For, the Angle V is in both the ſame; and the Angles at the Baſe differ only in this: That, in one, the External Angle; in the other, the Internal, (which is the others Complement to two Right-angles;) is equal to the Angle of C D at the Vertex.

XII. Hence it follows: That, *Of two Angles, where the Legs of the one are reſpectively equal to thoſe of the other; the one as much exceeding a Right-angle, as the other wants of it: The Square of the Baſe in the one, doth as much exceed the two Squares of the Legs; as, in the other, it wants thereof.*

XIII. And conſequently, *In any Right-lined Triangle, (however inclined,) the Squares of the Axis or Diameter, and of the half Baſes twice taken; are equal to the Squares of the Legs.* For, ſuppoſing C, C, the two halfs of the Baſe; and B, the Diameter or Axis of the Triangle, (meaning thereby a ſtreight Line from the Vertex to the middle of the Baſe;) and B, β, the two Legs: It is manifeſt, that, of the two Angles at the Baſe (which are each others Complement to two Right-angles;) the one doth as much exceed, as the other wants of, a Right-angle: And therefore the Square of one of the Legs, as Bq, doth as much exceed; as the other, βq, doth come ſhort of; $Dq + Cq$. And therefore, both together, $Bq + \beta q = 2Dq + 2Cq$.



XIV. And

XIV. And therefore, *The Baſe, and Axis (or Diameter) of a Triangle remaining the ſame; (however differently inclined:) the Aggregate of the Squares of the two Legs, remains the ſame.*

XV. And the ſame is to be underſtood of the *Squares of Tangents*, of a Parabola, Hyperbola, Eliplis, (or other Curve Line having Diameter and Ordinates,) *from the two ends of an Inſcribed Ordinate, to the Point of the Diameter (produced if need be) wherein thoſe Tangents meet.*

XVI. The ſame may be likewiſe accommodated, to the *Segments* (of ſuch Legs, or Tangents,) *Cut off by Lines Parallel to the Baſe.* Namely, *The Squares of ſuch Segments* (intercepted by thoſe Parallels) *together taken, (the Axe of ſuch Trapezium remaining the ſame,) are the ſame:* Whether ſuch Trapezium be Erect, or however inclined. For ſuch Segments, are ſtill Proportional to their Wholes.

CHAP. VII.

Application thereof to particular caſes.

I. IF A be a Right-angle, (or of 90 Degrees,) G r are Co-incident, and $\mu = o$. and therefore, $\frac{\mu}{G} CD = o$. And conſequently (by § 7, *Chap. preced.*) $Bq = (Cq + Dq \pm \frac{\mu}{G} CD =) Cq + Dq$.

II. If A = 120 Degrees; then is V (that is, the Angle contained of G r) = 60 Degrees: (as being always the Difference of 2 A from two Right-angles:) And conſequently G r μ an Equilater Triangle, (for ſuch alſo are the Angles at the Baſe; each of which is the Complement of A to two Right-angles:) And therefore, $\mu = G$; and $Bq = Cq + Dq + CD$.

III. If A = 60 Degrees: Then alſo is V = 60 Degrees, and $\mu = G$, as before. And therefore, $Bq = Cq + Dq - CD$.

IV. If A = 135. Then V = 90: And therefore, (by § 1.) $\mu q = Gq + rq$, that is (becauſe G = r,) $\mu q = 2Gq$; and $\mu = G\sqrt{2}$: And therefore, $Bq = Cq + Dq + CD\sqrt{2}$.

V. If A = 45. Then alſo, V = 90: And therefore, (as before) $\mu = G\sqrt{2}$; and conſequently, $Bq = Cq + Dq - CD\sqrt{2}$.

VI. If A = 150 } Then V = 120. And therefore, (by § 2.) $\mu q = Gq +$
VII. If A = 30 } $rq + Gr$; that is (becauſe G = r,) $\mu q = 3Gq$, and $\mu = G\sqrt{3}$.
And $Bq = Cq + Dq \pm CD\sqrt{3}$.

VIII. If A = 157½ } Then V = 135. And $\mu = G\sqrt{:2 + \sqrt{2}}$: (by § 4.) And
IX. If A = 22½ } therefore, $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{2}}$.

X. If A = 112½ } Then V = 45. And $\mu = G\sqrt{:2 - \sqrt{2}}$: (by § 5.) And
XI. If A = 6-½ } therefore, $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{2}}$.

XII. If

XII. If A = 165 } Then V = 150. And $\mu = G\sqrt{:2+\sqrt{3}:}$ (by § 6.) And there-
XIII. If A = 15 } fore, $Bq = Cq + Dq \pm (CD\sqrt{:2+\sqrt{3}:} =) \frac{\sqrt{6}+\sqrt{2}}{2} CD$.

XIV. If A = 105 } Then V = 30. And $\mu = G\sqrt{:2-\sqrt{3}:}$ (by § 7.) And there-
XV. If A = 75 } fore, $Bq = Cq + Dq \pm (CD\sqrt{:2-\sqrt{3}:} =) \frac{\sqrt{6}-\sqrt{2}}{2} CD$.

XVI. If $A = 172\frac{1}{2}$ } Then V = 165. And (by § 12.) $\mu = G\sqrt{:2+\sqrt{:2+\sqrt{3}}$.
XVII. If $A = 7\frac{1}{2}$ } $= G\sqrt{:2+\frac{\sqrt{6}+\sqrt{2}}{2}}$. And therefore, $Bq = Cq + Dq \pm$
$(CD\sqrt{:2+\sqrt{:2+\sqrt{3}}.} =) CD\sqrt{:2+\frac{\sqrt{6}+\sqrt{2}}{2}}$.

XVIII. If $A = 97\frac{1}{2}$ } Then V = 15. And (by § 13.) $\mu = G\sqrt{:2-\sqrt{:2+\sqrt{3}}$.
XIX. If $A = 82\frac{1}{2}$ } $= G\sqrt{:2-\frac{\sqrt{6}+\sqrt{2}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2-\sqrt{:2+\sqrt{3}}.=) CD\sqrt{:2-\frac{\sqrt{6}+\sqrt{2}}{2}}$.

XX. If $A = 142\frac{1}{2}$ } Then V = 105. And (by § 14.) $\mu = G\sqrt{:2+\sqrt{:2-\sqrt{3}}$.
XXI. If $A = 37\frac{1}{2}$ } $= G\sqrt{:2+\frac{\sqrt{6}-\sqrt{2}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2+\sqrt{:2-\sqrt{3}}.=) CD\sqrt{:2+\frac{\sqrt{6}-\sqrt{2}}{2}}$.

XXII. If $A = 127\frac{1}{2}$ } Then V = 75. And (by § 15.) $\mu = G\sqrt{:2-\sqrt{:2-\sqrt{3}}$.
XXIII. If $A = 52\frac{1}{2}$ } $= G\sqrt{:2-\frac{\sqrt{6}-\sqrt{2}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2-\sqrt{:2-\sqrt{3}}.=) CD\sqrt{:2-\frac{\sqrt{6}+\sqrt{2}}{2}}$.

And, in like manner, we may proceed to leſſer Arches, determined by quarters of Degrees. For like as here, by help of § 4, 5, 12, 13, 14, 15. we have performed § 8, 9, 10, 11, 16, 17, 18, 19, 20, 21, 22, 23. which proceed to half Degrees: So by the help of theſe, we may proceed to Quarters of Degrees. And farther if we pleaſe: But I ſhall at preſent reſt at half Degrees.

Moreover, aſſuming (as elſewhere proved) the Subtenſe of 36 Degrees, or the ſide of the inſcribed Decagon; Namely, $\frac{\sqrt{5}-1}{2}$ R. (by 9 *El.* 13. and 4 *El.* 14. Or, 55, 56, *Chap.* 32.) we may, from thence, thus proceed.

XXIV. If A = 108 } Then V = 36. And $\mu = \frac{\sqrt{5}-1}{2} G = G\sqrt{\frac{3-\sqrt{5}}{2}} = G\sqrt{:}$
XXV. If A = 72 } $2-\frac{\sqrt{5}+1}{2}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2-}$
$\frac{\sqrt{5}+1}{2}.=CD\sqrt{\frac{3-\sqrt{5}}{2}}=) \frac{\sqrt{5}-1}{2} CD$.

XXVI. If A = 144 } Then V = 108. And $\mu =$ (by § 24.) $G\sqrt{:2+\frac{\sqrt{5}-1}{2}}$.
XXVII. If A = 36 } $= G\sqrt{\frac{3+\sqrt{5}}{2}} = \frac{\sqrt{5}+1}{2} G$. And $Bq = Cq + Dq \pm$
$(CD\sqrt{:2+\frac{\sqrt{5}-1}{2}}.=CD\sqrt{\frac{3+\sqrt{5}}{2}}=) \frac{\sqrt{5}+1}{2} CD$.

XXVIII. If

XXVIII. If A = 126 } Then V = 72. And (by § 25.) $\mu = G\sqrt{:2 - \frac{\sqrt{5}-1}{2}}$
$= G\sqrt{\frac{5-\sqrt{5}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2 -}$
XXIX. If A = 54 } $\frac{\sqrt{5}-1}{2}$ =) $CD\sqrt{\frac{5-\sqrt{5}}{2}}$.

XXX. If A = 144 } Then V = 144. And (by § 26.) $\mu = G\sqrt{:2 + \frac{\sqrt{5}+1}{2}}$
$= G\sqrt{\frac{5+\sqrt{5}}{2}}$ And $Bq = Cq + Dq \pm CD\sqrt{}$
XXXI. If A = 18 } $\frac{5+\sqrt{5}}{2}$.

XXXII. If A = 153 } Then V = 126. And (by § 28.) $\mu = G\sqrt{:2 + \frac{\sqrt{5}-5}{2}}$.
XXXIII. If A = 27 } And $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{\frac{5-\sqrt{5}}{2}}}$.

XXXIV. If A = 117 } Then V = 54. And (by § 29.) $\mu = G\sqrt{:2 - \sqrt{\frac{5-\sqrt{5}}{2}}}$.
XXXV. If A = 63 } And $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{\frac{5-\sqrt{5}}{2}}}$.

XXXVI. If A = 171 } Then V = 162. And (by § 30.) $\mu = G\sqrt{:2 + \sqrt{\frac{5+\sqrt{5}}{2}}}$.
XXXVII. If A = 9 } And $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{\frac{5+\sqrt{5}}{2}}}$.

XXXVIII. If A = 99 } Then V = 18. And (by § 31.) $\mu = G\sqrt{:2 - \sqrt{\frac{5+\sqrt{5}}{2}}}$.
XXXIX. If A = 81 } And $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{\frac{5+\sqrt{5}}{2}}}$.

XL. If A = $166\frac{1}{2}$ } Then V = 153. And (by § 32.) $\mu = G\sqrt{:2 + \sqrt{:2 + \sqrt{}}}$
$\frac{5-\sqrt{5}}{2}$. And $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{:2 + \sqrt{}}}$
XLI. If A = $13\frac{1}{2}$ } $\frac{5-\sqrt{5}}{2}$.

XLII. If A = $103\frac{1}{2}$ } Then V = 27. And (by § 33.) $\mu = G\sqrt{:2 - \sqrt{:2 + \sqrt{}}}$
$\frac{5-\sqrt{5}}{2}$. And $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{:2 + \sqrt{}}}$
XLIII. If A = $76\frac{1}{2}$ } $\frac{5-\sqrt{5}}{2}$.

XLIV. If A = $148\frac{1}{2}$ } Then V = 117. And consequently, (by § 34.) Bq =
XLV. If A = $31\frac{1}{2}$ } $Cq + Dq \pm CD\sqrt{:2 + \sqrt{:2 - \sqrt{\frac{5-\sqrt{5}}{2}}}}$.

XLVI. If A = $121\frac{1}{2}$ } Then V = 63. And (by § 35.) $Bq = Cq + Dq \pm$
XLVII. If A = $58\frac{1}{2}$ } $CD\sqrt{:2 - \sqrt{:2 + \sqrt{\frac{5-\sqrt{5}}{2}}}}$.

XLVIII. If A = $175\frac{1}{2}$ } Then V = 171. And (by § 36.) $Bq = Cq + Dq \pm$
XLIX. If A = $4\frac{1}{2}$ } $CD\sqrt{:2+\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.

L. If A = $94\frac{1}{2}$ } Then V = 9. And (by § 37.) $Bq = Cq + Dq \pm CD\sqrt{:}$
LI. If A = $85\frac{1}{2}$ } $2-\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}$.

LII. If A = $139\frac{1}{2}$ } Then V = 99. And (by § 38.) $Bq = Cq + Dq \pm CD\sqrt{:}$
LIII. If A = $40\frac{1}{2}$ } $2+\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}$.

LIV. If A = $130\frac{1}{2}$ } Then V = 81. And (by § 39.) $Bq = Cq + Dq \pm CD\sqrt{:}$
LV. If A = $49\frac{1}{2}$ } $2-\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}$.

And, in like manner, by help of § 40, 41, *&c.* We may proceed to Arches determined by Quarters of Degrees; and further if need be.

Again, becauſe the Subtenſe of 72 Degrees, is $R\sqrt{\frac{5-\sqrt{5}}{2}}$ and the Subtenſe of 60 Degrees is R: We may thence Collect the Subtenſe of their Difference, which is that of 12 Degrees; namely, R into $\sqrt{\frac{15-3\sqrt{5}}{8}} - \sqrt{\frac{3+\sqrt{5}}{8}}$; or $\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}R$. And thence proceed thus,

LVI. If A = 96 } Then V = 12. And therefore, $Bq = Cq + Dq \pm$
LVII. If A = 84 } $\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}CD$.

LVIII. If A = 138 } Then V = 96. And (by § 56.) $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2+\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}: = CD\sqrt{\frac{7-\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4}}$
LIX. If A = 42 } $=) \frac{\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}CD$.

LX. If A = 132 } Then V = 84. And (by § 57.) $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2-\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4} = CD\sqrt{\frac{9+\sqrt{5}-\sqrt{:30-6\sqrt{5}:}}{4}}$
LXI. If A = 48 } $=) \frac{\sqrt{:30+6\sqrt{5}:}-\sqrt{5}+1}{4}CD$.

LXII. If A = 159 } Then V = 138. And (by § 58.) $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2+\sqrt{:2+\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}: = CD\sqrt{:2+\sqrt{}}$
LXIII. If A = 21 } $\frac{7-\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4} =) CD\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}$

LXIV. If A = 111 } Then V = 42. And (by § 59.) $Bq = Cq + Dq \pm (CD\sqrt{:}$
$2-\sqrt{:2+\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}: = CD\sqrt{:2-\sqrt{}}$
LXV. If A = 69 } $\frac{7-\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4} =) CD\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}$

LXVI. If

LXVI. If A = 156 } Then V = 132. And (by § 60.) $Bq = Cq + Dq \pm (CD\sqrt{:} 2 + \sqrt{:} 2 - \frac{\sqrt{:}30 - 6\sqrt{5:} - \sqrt{5} - 1}{4} :) = CD\sqrt{:} 2 + \sqrt{\frac{8 - \sqrt{:}30 - 6\sqrt{5:} + \sqrt{5} + 1}{4}} = CD\sqrt{:} 2 + \sqrt{}$

LXVII. If A = 24 } $\frac{9 + \sqrt{5} - \sqrt{:}30 - 6\sqrt{5:}}{4} = CD\sqrt{\frac{8 + \sqrt{:}30 + 6\sqrt{5:} - \sqrt{5} + 1}{4}} = CD\sqrt{\frac{9 - \sqrt{5} + \sqrt{:}30 + 6\sqrt{5:}}{4}} =) \frac{\sqrt{:}30 - 6\sqrt{5:} + \sqrt{5} + 1}{4} CD.$

LXVIII. If A = 114 } Then V = 48. And (by § 61.) $Bq = Cq + Dq \pm (CD\sqrt{:} 2 - \sqrt{:} 2 - \frac{\sqrt{:}30 - 6\sqrt{5:} - \sqrt{5} - 1}{4} :) = CD\sqrt{:} 2 - \sqrt{\frac{8 - \sqrt{:}30 - 6\sqrt{5:} + \sqrt{5} + 1}{4}} = CD\sqrt{:} 2 - \sqrt{}$

LXIX. If A = 66 } $\frac{9 + \sqrt{5} - \sqrt{:}30 - 6\sqrt{5:}}{4} = CD\sqrt{:} 2 - \frac{\sqrt{:}30 + 6\sqrt{5:} - \sqrt{5} + 1}{4} = CD\sqrt{\frac{7 + \sqrt{5} - \sqrt{:}30 + 6\sqrt{5:}}{4}} =) \frac{\sqrt{15} + \sqrt{3} - \sqrt{:}10 - 2\sqrt{5:}}{4} CD.$

LXX. If A = $169\frac{1}{2}$ } Then V = 159. And (by § 62.) $Bq = Cq + Dq \pm$

LXXI. If A = $10\frac{1}{2}$ } $CD\sqrt{:} 2 + \sqrt{\frac{8 + \sqrt{15} - \sqrt{3} + \sqrt{:}10 + 2\sqrt{5:}}{4}}$.

LXXII. If A = $100\frac{1}{2}$ } Then V = 21. And (by § 63.) $Bq = Cq + Dq \pm$

LXXIII. If A = $79\frac{1}{2}$ } $CD\sqrt{:} 2 - \sqrt{\frac{8 + \sqrt{15} - \sqrt{3} + \sqrt{:}10 + \sqrt{2}\sqrt{5:}}{4}}$.

LXXIV. If A = $145\frac{1}{2}$ } Then V = 111. And (by § 64.) $Bq = Cq + Dq \pm$

LXXV. If A = $34\frac{1}{2}$ } $CD\sqrt{:} 2 + \sqrt{\frac{8 - \sqrt{15} + \sqrt{3} - \sqrt{:}10 + 2\sqrt{5:}}{4}}$.

LXXVI. If A = $124\frac{1}{2}$ } Then V = 69. And (by § 65.) $Bq = Cq + Dq \pm$

LXXVII. If A = 55 } $CD\sqrt{:} 2 - \sqrt{\frac{8 - \sqrt{15} + \sqrt{3} - \sqrt{:}10 + 2\sqrt{5:}}{4}}$.

LXXVIII. If A = 168 } Then V = 156. And (by § 66.) $Bq = Cq + Dq \pm (CD\sqrt{:} 2 + \frac{\sqrt{:}30 - 6\sqrt{5:} + \sqrt{5} + 1:}{4} = CD\sqrt{\frac{9 + \sqrt{5} + \sqrt{:}30 - 6\sqrt{5:}}{4}} =) \frac{\sqrt{:}30 + 6\sqrt{5:} + \sqrt{5} - 1}{4} CD.$

LXXIX. If A = 12 }

LXXX. If A = 102 } Then V = 24. And (by § 67.) $Bq = Cq + Dq \pm (CD\sqrt{:} 2 - \frac{\sqrt{:}30 - 6\sqrt{5:} + \sqrt{5} + 1:}{4} = CD\sqrt{\frac{7 - \sqrt{5} - \sqrt{:}30 - 6\sqrt{5:}}{4}} =) \frac{\sqrt{:}10 + 2\sqrt{5:} - \sqrt{15} + \sqrt{3}}{4} CD.$

LXXXI. If A = 78 }

LXXXII. If A = 147 } Then V = 114. And (by § 68.) $Bq = Cq + Dq \pm (CD\sqrt{:} 2 + \frac{\sqrt{15} + \sqrt{3} - \sqrt{:}10 - 2\sqrt{5:}}{4} =) CD\sqrt{\frac{8 + \sqrt{15} + \sqrt{3} - \sqrt{:}10 - 2\sqrt{5:}}{4}}$.

LXXXIII. If A = 33 }

LXXXIV. If A = 123 } Then V = 66. And (by § 69.) $Bq = Cq + Dq \pm (CD\sqrt{:}\, 2 - \frac{\sqrt{15} + \sqrt{3} - \sqrt{:10 - 2\sqrt{5}:}}{4}) \; CD\sqrt{}$

LXXXV. If A = 57 } $\frac{8 - \sqrt{15} - \sqrt{3} + \sqrt{:10 - 2\sqrt{5}:}}{4}$.

LXXXVI. If A = 174 } Then V = 168. And (by § 78.) $Bq = Cq + Dq \pm (CD\sqrt{:}\, 2 + \frac{\sqrt{:30 + 6\sqrt{5}:} + \sqrt{5} - 1:}{4} = CD\sqrt{} \frac{7 + \sqrt{5} + \sqrt{:30 + 6\sqrt{5}:}}{4}$

LXXXVII. If A = 6 } $=) \frac{\sqrt{15} + \sqrt{3} + \sqrt{:10 - 2\sqrt{5}:}}{4} CD.$

LXXXVIII. If A = 141 } Then V = 102. And (by § 80.) $Bq = Cq + Dq \pm (CD\sqrt{:}\, 2 + \frac{\sqrt{:10 + 2\sqrt{5}:} - \sqrt{15} + \sqrt{3}}{4}) \; CD\sqrt{}$

LXXXIX. If A = 39 } $\frac{8 + \sqrt{3} - \sqrt{15} + \sqrt{:10 + 2\sqrt{5}:}}{4}$.

XC. If A = 129. } Then V = 78. And (by § 81.) $Bq = Cq + Dq \pm (CD\sqrt{:}\, 2 - \frac{\sqrt{:10 + 2\sqrt{5}:} - \sqrt{15} + \sqrt{3}:}{4}) \; CD\sqrt{}$

XCI. If A = 51 } $\frac{8 - \sqrt{3} + \sqrt{15} - \sqrt{:10 + 2\sqrt{5}:}}{4}$.

XCII. If A = $163\frac{1}{2}$ } Then V = 147. And (by § 82.) $Bq = Cq + Dq \pm CD\sqrt{:}\, 2 + \sqrt{\frac{8 + \sqrt{3} + \sqrt{15} - \sqrt{:10 - 2\sqrt{5}:}}{4}}$.

XCIII. If A = $16\frac{1}{2}$ }

XCIV. If A = $106\frac{1}{2}$ } Then V = 33. And (by § 83.) $Bq = Cq + Dq \pm CD\sqrt{:}\, 2 - \sqrt{\frac{8 + \sqrt{3} + \sqrt{15} - \sqrt{:10 - 2\sqrt{5}:}}{4}}$.

XCV. If A = $73\frac{1}{2}$ }

XCVI. If A = $151\frac{1}{2}$ } Then V = 123. And (by § 84.) $Bq = Cq + Dq \pm CD\sqrt{:}\, 2 + \sqrt{\frac{8 - \sqrt{3} - \sqrt{15} + \sqrt{:10 - 2\sqrt{5}:}}{4}}$.

XCVII. If A = $28\frac{1}{2}$ }

XCVIII. If A = $118\frac{1}{2}$ } Then V = 57. And (by § 85.) $Bq = Cq + Dq \pm CD\sqrt{:}\, 2 + \sqrt{\frac{8 - \sqrt{3} - \sqrt{15} + \sqrt{:10 - 2\sqrt{5}:}}{4}}$.

XCIX. If A = $61\frac{1}{2}$ }

C. If A = 177 } Then V = 174. And (by § 86.) $Bq = Cq + Dq \pm (CD\sqrt{:}\, 2 + \frac{\sqrt{15} + \sqrt{3} + \sqrt{:10 - 2\sqrt{5}:}}{4}) \; CD\sqrt{}$

CI. If A = 3 } $\frac{8 + \sqrt{3} + \sqrt{15} + \sqrt{:10 - 2\sqrt{5}:}}{4}$.

CII. If A = 93 } Then V = 6. And (by § 87.) $Bq = Cq + Dq \pm (CD\sqrt{:}\, 2 - \frac{\sqrt{15} + \sqrt{3} + \sqrt{:10 - 2\sqrt{5}:}}{4}) \; CD\sqrt{}$

CIII. If A = 87 } $\frac{8 - \sqrt{3} - \sqrt{15} - \sqrt{:10 - 2\sqrt{5}:}}{4}$.

CIV. If A = $160\frac{1}{2}$ } Then V = 141. And (by § 88.) $Bq = Cq + Dq \pm CD\sqrt{:}\, 2 + \sqrt{\frac{8 + \sqrt{3} - \sqrt{15} + \sqrt{:10 + 2\sqrt{5}:}}{4}}$.

CV. If A = $19\frac{1}{2}$ }

CVI. If

CVI. If A = $109\frac{1}{2}$ } Then V = 39. And (by § 89.) $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{\frac{8+\sqrt{3}-\sqrt{15}+\sqrt{:10+2\sqrt{5}:}}{4}}}$.
CVII. If A = $70\frac{1}{2}$ }

CVIII. If A = $154\frac{1}{2}$ } Then V = 129. And (by § 90.) $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{\frac{8-\sqrt{3}+\sqrt{15}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.
CIX. If A = $25\frac{1}{2}$ }

CX. If A = $115\frac{1}{2}$ } Then V = 51. And (by § 91.) $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{\frac{8-\sqrt{3}+\sqrt{15}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.
CXI. If A = $64\frac{1}{2}$ }

CXII. If A = $178\frac{1}{2}$ } Then V = 177. And (by § 100.) $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{\frac{8+\sqrt{3}+\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.
CXIII. If A = $1\frac{1}{2}$ }

CXIV. If A = $91\frac{1}{2}$ } Then V = 3. And (by § 101.) $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{\frac{8+\sqrt{3}+\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.
CXV. If A = $88\frac{1}{2}$ }

CXVI. If A = $136\frac{1}{2}$ } Then V = 93. And (by § 102.) $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{\frac{8-\sqrt{3}-\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.
CVII. If A = $43\frac{1}{2}$ }

CXVIII. If A = $133\frac{1}{2}$ } Then V = 87. And (by § 107.) $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{\frac{8-\sqrt{3}-\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.
CXIX. If A = $46\frac{1}{2}$ }

And, in like manner, (by help of § 70, 71, *&c.* 92, 93, *&c.* 104, 105, *&c.* as was ſhewed at § 23.) we may proceed to Arches determined by Quarters of Degrees; or yet further, if there be occaſion.

But we content our ſelves at preſent to reſt at half Degrees. Having hereby fitted ſubtenſes to every three halves of a Degree throughout the Semicircle.

CHAP.

CHAP. VIII.

Of the Canon of Subtenſes, and Sines; Of Tangents alſo and of Secants.

FROM what is delivered in the foregoing Chapter; it is eaſie to conſtruct a Canon of Subtenſes or Chords, in Surd Roots, to every Three-halves of a Degree throughout the Semicircle. The halves of which Subtenſes, are the Right-ſines for every Three-quarters of a Degree throughout the Quadrant.

(And thence, if need be, many Canons of Tangents and Secants, be deduced, in Surds Roots.)

And hereby, any who pleaſe, may either make new Tables, in Numbers, (to what accuracy he pleaſe,) or examin thoſe already made.

For to every Subtenſe, to be ſucceſſively ſought, there will need but one extraction of the Square Root; (and, ſometimes, not this;) the reſt of the work being diſpatched by only Addition and Subtraction; or, at moſt, Diviſion alſo by 2 or 4.

As, for Example: Suppoſing the Radius of a Circle $R = 1$. Then (becauſe theſe, in the ſame Circle be all equal,) $C = D = 1$. And likewiſe $Cq = Dq = CD = 1$. And B will be the Subtenſe of the Angle propoſed.

Therefore, (by § 1.) the Square of the Subtenſe of 90 Degrees, $Bq = Cq + Dq = 1 + 1 = 2$. And the Subtenſe it ſelf $B = \sqrt{2}$: Which is had by one extraction of the Square Root of the number 2; continued in Decimal Parts to to what accuracy we pleaſe. Suppoſe $\sqrt{2} = 1.42421356 +$ *proximè*.

Again, (by § 2.) the Square of the Subtenſe of 120 Degrees, is $Bq = Cq + Dq + CD = 1 + 1 + 1 = 3$. And the Subtenſe it ſelf $B = \sqrt{3}$: Which is likewiſe had by one extraction of the Square Root of 3. Suppoſe $\sqrt{3} = 1.7320508 1 —$ *proximè*.

The Square of the Subtenſe of 60 Degrees, is (by § 3.) $Bq = Cq + Dq - CD = 1 + 1 - 1 = 1$. And therefore the Subtenſe $B = 1$.

The Square of the Subtenſe of 135 Degrees, is (by § 4.) $Bq = Cq + Dq + CD\sqrt{2} = 1 + 1 + \sqrt{2} = 2 + \sqrt{2}$. Which is had by adding 2, to the value of $\sqrt{2}$ already found at § 1. That is, $2 + \sqrt{2} = 3.41421356 +$. And ſo by one extraction of the Square Root of this number, we have the Subtenſe of 135 Degrees: Namely, $B = (\sqrt{:2 + \sqrt{2}:} = \sqrt{3.41421356 + } =) 1.84775906\frac{1}{2}$ *proximè*.

So (by § 5.) the Square of the Subtenſe of 45 Degrees, is $Bq = Cq + Dq - CD\sqrt{2} = 1 + 1 - \sqrt{2} = 2 - \sqrt{2}$. That is (by ſubtracting from 2 the value of $\sqrt{2}$ already found,) $Bq = 0.58578643\frac{1}{2}$ *proximè*. The Square Root of which, now to be extracted, is $B = \sqrt{:2 - \sqrt{2}:} = 0.76536686\frac{2}{3}$ *proximè*.

And

And (by § 6.) the Square of the Subtenſe of 150 Degrees, is $Bq+Cq+CD\sqrt{3}=1+1+\sqrt{3}=2+\sqrt{3}$. Which is had, by adding 2 to the value of $\sqrt{3}$ all ready found. That is, $Bq=2+\sqrt{3}=3.73205081-$. The Root of which (now to be extracted) is $B=\sqrt{:2+\sqrt{3}:}=1.93185165+$ *proximè*.

Or thus: Becauſe $\sqrt{:2+\sqrt{3}:}=\frac{\sqrt{6}+\sqrt{2}}{2}$, (as will appear, either by the Squaring of this, or by extracting the Square Root of the Binomial $2+\sqrt{3}$:) Having, as before, the value of $\sqrt{2}=1.41421356+$; and (by one extraction now to be made) the value of $\sqrt{6}=2.44948974+$ (or, it may be had by Multiplying the value of $\sqrt{2}$, by that of $\sqrt{3}$, already known; becauſe $\sqrt{2}\times\sqrt{3}=\sqrt{6}$:) we have thence $\sqrt{6}+\sqrt{2}=3.86370330+$; and the half thereof $\frac{\sqrt{6}+\sqrt{2}}{2}=1.93185165+$. As before,

So (by § 7.) the Square of the Subtenſe of 30 Degrees, is $Bq=Cq+Dq-CD\sqrt{3}=1+1-\sqrt{3}=2-\sqrt{3}=0.26794919-$; (which is had by Subduction only, the value of $\sqrt{3}$ being found before.) The Square Root of which (now to be extracted) is the Subtenſe $B=\sqrt{:2-\sqrt{3}:}=0.5176380\text{9}$ *proximè*.

Or thus, (without extracting a Root;) becauſe $\sqrt{:2-\sqrt{3}:}=\frac{\sqrt{6}-\sqrt{2}}{2}$: Therefore, (the values of $\sqrt{6}$ and $\sqrt{2}$ being had before:) by Subduction only we have $\sqrt{6}-\sqrt{2}=1.03527618$ *proximè*; and (the half of this) $\frac{\sqrt{6}-\sqrt{2}}{2}=0.5176380\text{9}$ *proximè*: As before,

And in the reſt, (taking the Propoſitions or Paragraphs as they are before ſet down in the former Chapter,) there is need but of one extraction of the Square Root (and oft-times not of one,) for finding of each Subtenſe.

Theſe Subtenſes being thus had; the halves thereof are the Right-ſines of the half Arch. As for Example.

Arches.	Subtenſes.	Sines.	Arches.
Degrees. 90	1.41421356+	0.7071067 8+	45 Degrees.
120	1.73205081-	0.86602540 $\frac{1}{2}$	60
60	1.00000000	0.50000000	30
135	1.84775906 $\frac{1}{2}$	0.92387953 $\frac{1}{4}$	67 $\frac{1}{2}$
45	0.76536687-	0.38268343 $\frac{1}{2}$	22 $\frac{1}{2}$
150	1.93185165+	0.96592582 $\frac{1}{2}$	75
30	0.51763809	0.25881904 $\frac{1}{2}$	15

Now follows the Table of Subtenſes in Surd Roots, anſwering to each three halves of a Degree throughout the whole Semicircle; (and conſequently, of their Reſiduals to a whole Circle, whoſe Subtenſes are the ſame with theſe:) Putting the Radius $R=1$, and therefore $C=D=1$, and likewiſe $Cq=Dq=CD=1$: With references in the Margin to the Paragraphs of the former Chapter from whence they are derived.

§	Deg.	Subtenses.
	0	0.
113	$1\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
101	3	$\sqrt{:2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.
49	$4\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
87	6	$\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}$.
17	$7\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{3}}}}=\sqrt{:2-\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{2}}}$.
37	9	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.
71	$10\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
79	12	$\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}$.
41	$13\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
13	15	$\sqrt{:2-\sqrt{:2+\sqrt{3}}}=\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}$.
93	$16\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
31	18	$\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}=\frac{\sqrt{10}+\sqrt{2}-2\sqrt{:5-\sqrt{5}:}}{4}$.
105	$19\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
63	21	$\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}$.
9	$22\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{2}}}$.
67	24	$\frac{\sqrt{:10+2\sqrt{5}:}-\sqrt{15}+\sqrt{3}}{4}$.
109	$25\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
33	27	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$.
97	$28\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
7	30	$\sqrt{:2-\sqrt{3}:}=\frac{\sqrt{6}-\sqrt{2}}{2}$.
45	$31\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
83	33	$\sqrt{:2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.
75	$34\frac{2}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
27	36	$\sqrt{\frac{3-\sqrt{5}}{2}}=\frac{\sqrt{5}-1}{2}$.
21	$37\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2-\sqrt{3}}}}=\sqrt{:2-\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}}$.
89	39	$\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}$.
53	$40\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
59	42	$\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}$.
117	$43\frac{1}{2}$	$\sqrt{:2-\sqrt{:2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
5	45	$\sqrt{:2-\sqrt{2}}$.

§	Deg. 180	Subtenses. 2.
112	$178\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
100	177	$\sqrt{:2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.
48	$175\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
86	174	$\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}$.
16	$172\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}=\sqrt{:2+\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{2}}}$.
36	171	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.
70	$169\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
78	168	$\frac{\sqrt{:15+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}}{4}$.
40	$166\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
12	165	$\sqrt{:2+\sqrt{:2+\sqrt{3}}}=\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{2}}$.
92	$163\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
30	162	$\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}.=\frac{\sqrt{10}+\sqrt{2}+2\sqrt{:5-\sqrt{5}:}}{4}$.
104	$160\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
62	159	$\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}$.
8	$157\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{2}}}$.
66	156	$\frac{\sqrt{:30+6\sqrt{5}:}+\sqrt{5}-\sqrt{1}}{4}$.
108	$154\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
32	153	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$.
96	$151\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
6	150	$\sqrt{:2+\sqrt{3}}.=\frac{\sqrt{6}+\sqrt{2}}{2}$.
44	$148\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
82	147	$\sqrt{:2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.
74	$145\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
26	144	$\sqrt{\frac{5+\sqrt{5}}{2}}$.
20	$142\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{:2-\sqrt{3}}}}=\sqrt{:2+\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}}$.
88	141	$\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}$.
52	$139\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
58	138	$\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}$.
116	$136\frac{1}{2}$	$\sqrt{:2+\sqrt{:2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
4	135	$\sqrt{:2+\sqrt{2}}$.

§	Deg.	Subtenſes.
5	45	$\sqrt{:2-\sqrt{2}:}$
119	$46\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
61	48	$\frac{\sqrt{15}+\sqrt{3}-\sqrt{:10-\sqrt{25}:}}{4}$
55	$49\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
91	51	$\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.
23	$52\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{:2-\sqrt{3}}}}=\sqrt{:2-\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}}$.
29	54	$\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}$.
77	$55\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$
85	57	$\sqrt{:2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.
47	$58\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
3	60	1.
99	$61\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$
35	63	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}$.
111	$64\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$
69	66	$\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}$.
11	$67\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{2}}}$.
65	69	$\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.
107	$70\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
25	72	$\sqrt{\frac{5-\sqrt{5}}{2}}$.
95	$73\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
15	75	$\sqrt{:2-\sqrt{:2-\sqrt{3}}}=\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}$.
43	$76\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
81	78	$\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}$.
73	$79\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
39	81	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}$.
19	$82\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{:2+\sqrt{3}}}}=\sqrt{:2-\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}}$.
57	84	$\frac{\sqrt{:30+6\sqrt{5}:}-\sqrt{5}+1}{4}$.
51	$85\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
103	87	$\sqrt{:2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.
115	$88\frac{1}{2}$	$\sqrt{:2-\sqrt{:2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
1	90	$\sqrt{2}$.

§	Deg.	Subtenſes.
4	135	$\sqrt{:2+\sqrt{2}}$.
118	133½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
60	132	$\frac{\sqrt{:30-6\sqrt{5}:}+\sqrt{5}+1}{4}$
54	130½	$\sqrt{:2+\sqrt{:2-\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
90	129	$\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.
22	127½	$\sqrt{:2+\sqrt{:2-\sqrt{:2-\sqrt{3}}}}=\sqrt{:2+\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}}$.
28	126	$\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}$.
76	124½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$
84	123	$\sqrt{:2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{2}}}$.
46	121½	$\sqrt{:2+\sqrt{:2-\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
2	120	$\sqrt{3}$.
98	118½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$
34	117	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}$.
110	115½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}}$
68	114	$\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}$.
10	112½	$\sqrt{:2+\sqrt{:2-\sqrt{2}}}$.
64	111	$\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.
106	109½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
24	108	$\sqrt{\frac{3+\sqrt{5}}{2}}=\frac{\sqrt{5}+1}{2}$.
94	106½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
14	105	$\sqrt{:2+\sqrt{:2-\sqrt{3}}}=\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}$.
42	103½	$\sqrt{:2+\sqrt{:2-\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
80	102	$\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}$.
72	100½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}}$.
38	99	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}$.
48	97½	$\sqrt{:2+\sqrt{:2-\sqrt{:2+\sqrt{3}}}}=\sqrt{:2+\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}}$.
56	96	$\frac{\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}$.
50	94½	$\sqrt{:2+\sqrt{:2-\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
102	93	$\sqrt{:2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.
114	91½	$\sqrt{:2+\sqrt{:2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}}}$.
1	90	$\sqrt{2}$.

And, in like manner, we may proceed to deſign, by Surd Roots, the Subtenſes of Arches as ſmall as we pleaſe, by a continual Biſection of theſe Arches. The halves of which Subtenſes, are the Right Sines of the Half-Arches.

But, to deſign an intire Canon of Subtenſes and Sines, anſwering to each ſingle Degree, and the Sexageſims or firſt Minutes of ſuch Degrees: Will (beſide the extracting the Square Roots, of ſuch Surds, in Numbers,) require alſo the Analyſis (in Numbers) of Two Triſections, and of one Quinquiſection of an Arch.

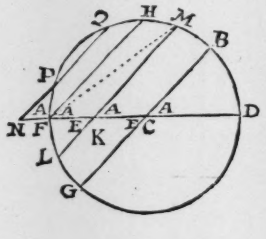
For, the former proceſs reaching no farther than to the Subtenſe of $1\frac{1}{2}$ Degree; and conſequently to the ſine of $\frac{1}{4}$ of a Degree, or of Min. $45 = 3 \times 3 \times 5$: We may thence, by a Triſection twice performed; and a Quinquiſection once, proceed to the ſine of 1 Minute. But not by Biſections only, or operations thence deduced.

But, theſe operations being ſo (as is ſaid) performed; the reſt of the work is eaſily diſpatched by help of § 9, 10. *Chap.* 6. for finding the Subtenſe of the Sum or Difference of thoſe Arches whoſe Subtenſes are already known.

CHAP. IX.

Of Angles compared with they Arches on which the ſtand.

I. THAT *The Angle of a Sector, is proportional to the Arch on which it doth inſiſt; whether ſuch Angle be at the Center, or at the Circumference:* And, that *ſuch Angle at the Center, is double to that at the Circumference:* Is ſhewed by *Euclide* long ſince; and is generally known. But not ſo, in caſe ſuch Angle be any where elſe, whether within or without the Circle. It will therefore be not amiſs, to purſue that Notion a little farther; as here followeth.



II. If on the Diameter of a Circle D F, be formed, at the Center C, an Angle B A D; the intercepted Arch B D, is Proportional to the Angle. (by 26, 27. *El.* 3. and 33, *El.* 6, of *Euclid.*) That is, the Arch intercepted B D, is ſuch a Part of the whole Circumference; as is the Angle A, of Four Right-angles. And accordingly the Angle A, is ſaid to be, of ſo many Degrees, as is the Arch B D.

III. If B A, D A, the Legs containing ſuch Angle, after a Decuſſation at C, forming the Vertical Angle E, be continued, on the other ſide, to the Circumference; the intercepted Arch G F, will, (by § 2.) be equal to B D, (becauſe of the Vertical Angle E, equal to A:) And, conſequently, the Aggregate of both B D + G F, is double to B D. That is, B D + G F = 2 B D.

IV. If at F the end of the Diameter, be formed a like Angle A; the intercepted Arch H D, is likewiſe double to B D. That is, H D = B D + G F = 2 B D. Becauſe (by 20. *El.* 3.) the Angle at the Center, is double to that at the Circumference, on the ſame Arch.

Or

Or thus, becauſe (by Conſtruction) F H, G C B, are Parallels: (as making equal Angles with F D:) Therefore, H B = F G = B D. And, conſequently, H D = H B + B D = F G + B D = 2 B D. (By § 3.)

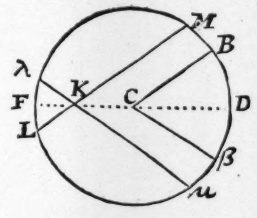
V. If at K, (any other Point of the Diameter within the Circle,) ſuch Angle A be made, (with its Vertical E:) the Aggregate of the two Arches intercepted, F L + M D, is Double to B D. That is, F L + M D = F G + B D = 2 B D. For, (drawing the ſtreight Line A M,) the two Internal Angles F M L + M F D, are equal to the External M K D = A = H F D (which is an Angle at the Circumference.) And therefore, the Arches oppoſite to thoſe F L + M D, equal to the Arch oppoſite to this H D = 2 B D. (By § 4.)

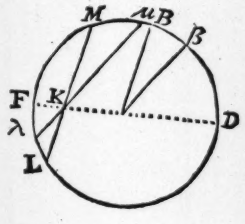
Or thus, becauſe F H, L M, be Parallels (as making like Angles with F D,) therefore, F L = H M, and F L + M D = H M + M D = H D = 2 B D. (By § 4.)

VI. If at N, a Point of the Diameter produced, without the Circle; be formed a like Angle A; the Difference of the two intercepted Arches, Q D — P F, is equal to the ſame H D, or the Double of B D. For, P Q, F H, being Parallels, (as making like Angles with D F produced,) and therefore, P F = Q H: Therefore, Q D — P F = Q D — Q H = H D = 2 B D. (By § 4.)

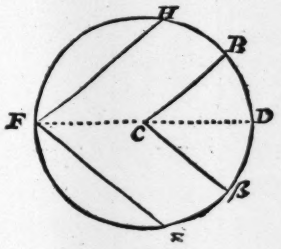
VII. The ſame will hold, though neither of the Legs containing the Angle do paſs through the Center, (and therefore lie not upon a Diameter;) as I ſhall now ſhew in the ſeveral Caſes.

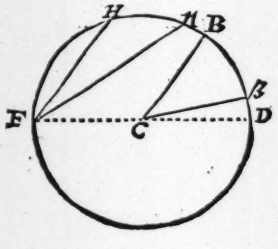
VIII. If M L, $\mu\lambda$, make M K μ an Angle at K any where, within the Circle, Let B c β be a like Angle at the Center, and the Legs of this Parallel to thoſe of that. And, by the Angular Point K, draw the Diameter F K D. Then is (by § 5.) M D + L F = 2 B D: And μ D + λ F = 2 β D. Therefore, (the Sum or Difference) M μ (= M D $\pm$ μ D) + L λ (= L F $\pm$ λ F) = (2 B D $\pm$ 2 β D =) 2 B β.





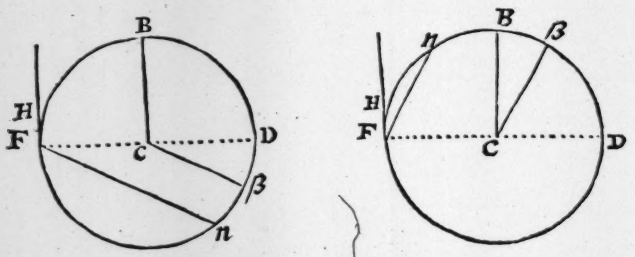
IX. In like manner: If H F ν be an Angle at the Circumference: And, at the Center, B C β like to it, and with Legs Parallel to thoſe: And F D a Diameter. Then (by § 4.) H D = 2 B D, and ν D = 2 β D. Therefore, H ν (= H D $\pm$ ν D) = (2 B D $\pm$ 2 β D =) 2 B β.



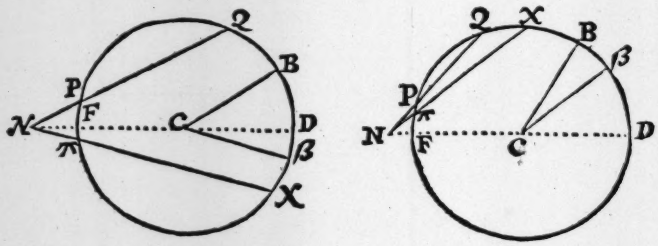


X. The

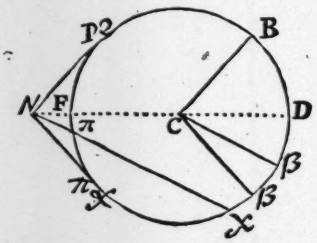
X. The ſame is to be underſtood, in caſe one of the Legs touch the Circumference at F, (the Points F, H, being in this caſe Co-incident; the Arch FH vaniſhing to nothing, and the Arch intercepted Hη, the ſame with Fη.) For here alſo, HFD = 2BD, and ηFD = 2βD; and therefore, HFη (= HFD ± ηFD) = (2BD ± 2βD =) 2Bβ.



XI. In like manner: If QNχ be an Angle without the Circle, whoſe Legs cut it in P, π: And, at the Center, a like Angle and like ſited BC : And NFD a Diameter produced. Then (by § 6.) QD − PF = 2BD, and χD − πF = 2βD. And therefore QD ± χD *wanting* PF ± πF, that is Qχ − Pπ, is equal to 2BD ± 2βD = 2Bβ.



XII. The ſame is to be underſtood, in caſe one or both of the Legs do not cut, but only touch the Circle. For then the Points P, Q; or π, χ; (or both,) being Co-incident; the reſt proceeds as before. For ſtill QD − PF = 2BD, and χD − πF = 2βD; and therefore, (the Sum or Difference) Qχ − Pπ = 2Bβ.



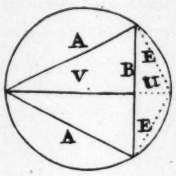
XIII. But in caſe one or both of the Legs paſs by the Circle, and neither cut, nor ſo much as touch it: It doth not concern the preſent buſineſs; for ſuch Angle doth not inſiſt on a Circular Arch. The whole therefore (thus demonſtrated) amounts to this General.

XIV. *If a Circle be cut* (or at leaſt touched) *by two ſtreight Lines, making an Angle:* (and ſo, when continued, interſecting each other:) *The Sum* (if their interſection be within the Circle,) *or Difference* (if without,) *of the two Arches intercepted by them* (produced, if need be,) *or* (if their interſection be at the Circumference) *the ſingle Arch by them intercepted; is Double to the Arch of a like Angle at the Center.*

FINIS.

Angular Sections.

I



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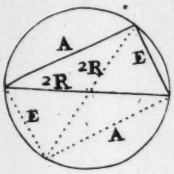

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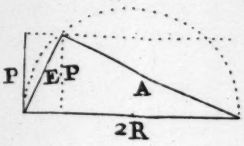

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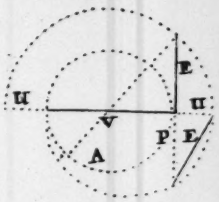
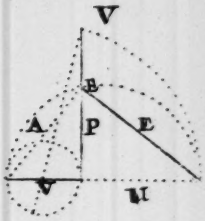


V



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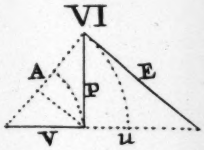

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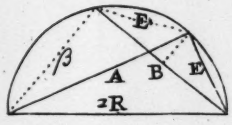
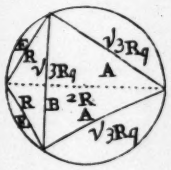


VIII



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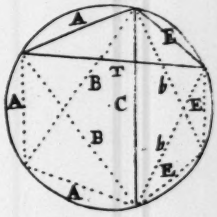

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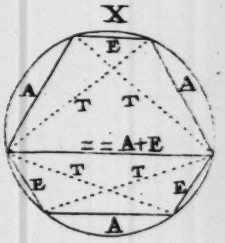

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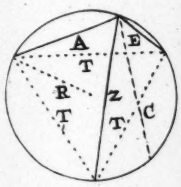
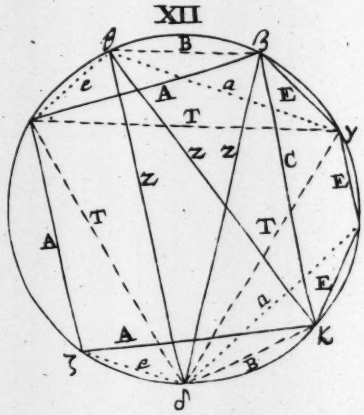


XII



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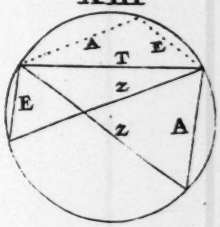

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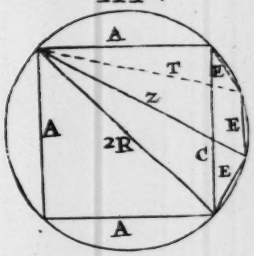

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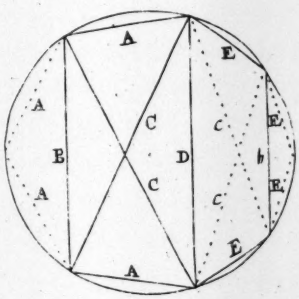

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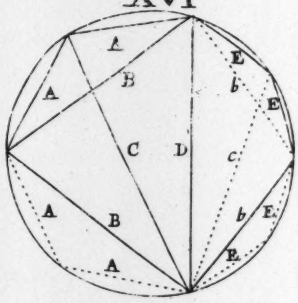

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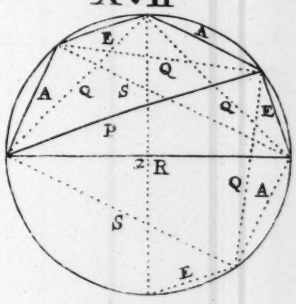

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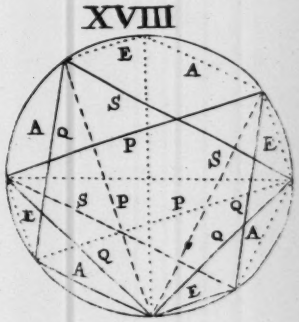

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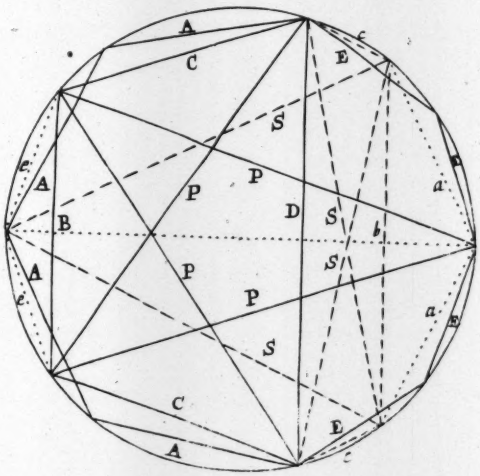

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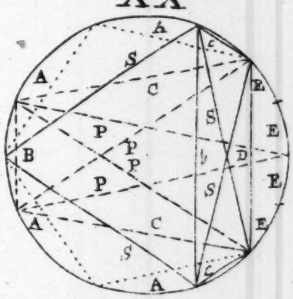

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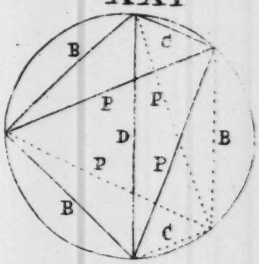

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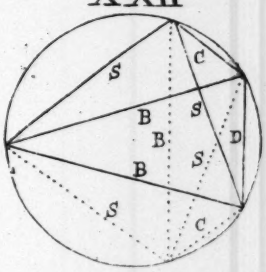
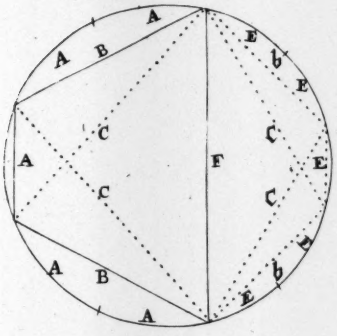


XXIII

Angular Sections

XXIV



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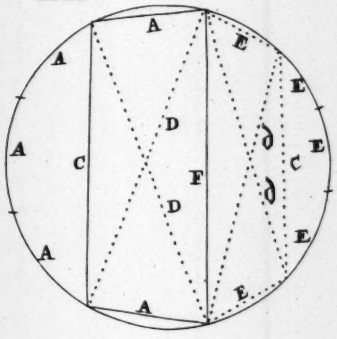

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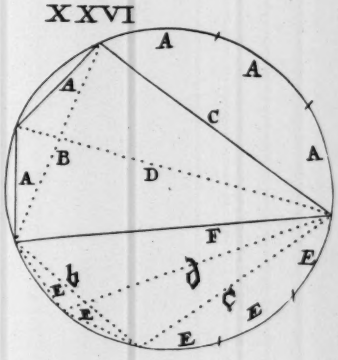

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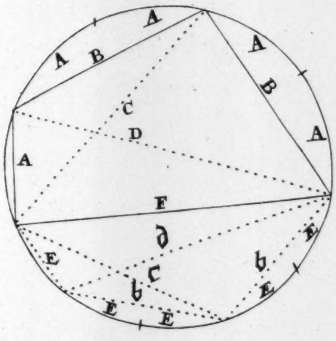

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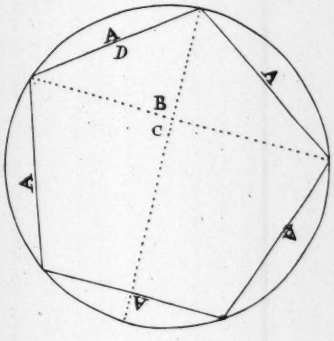

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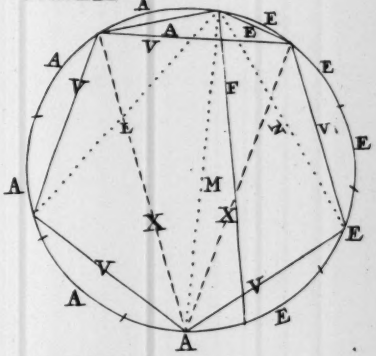

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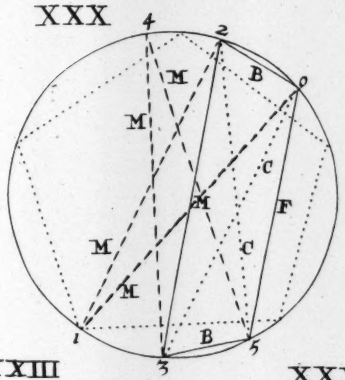

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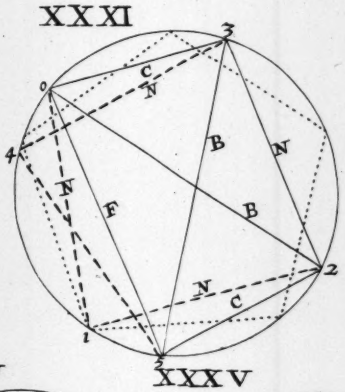

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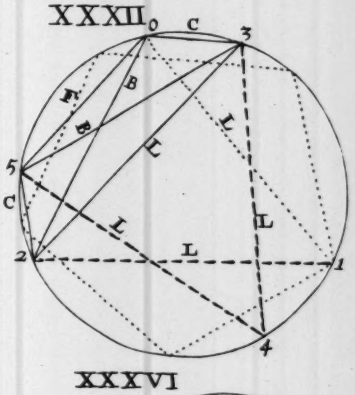

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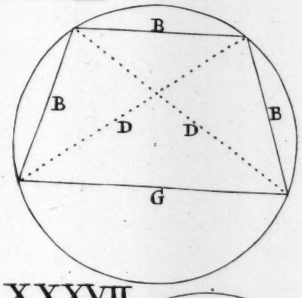

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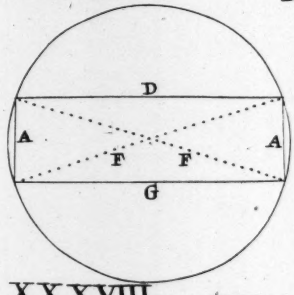

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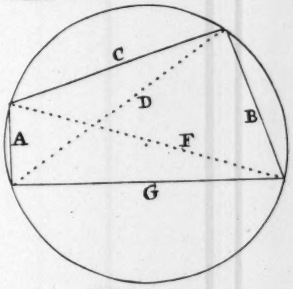

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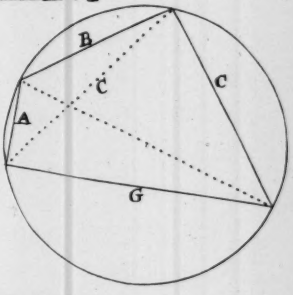

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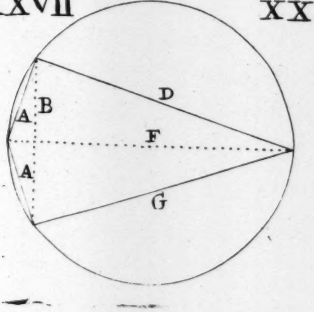

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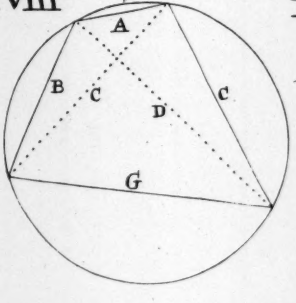
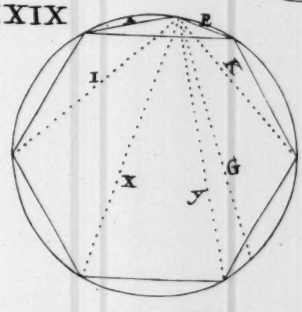
XXXIX